Lecture 13. Friable integers. The saddle point method. Integers free of small prime factors

M. Klazar

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In the thirteenth lecture we cover the last two Chapters III.5. *Friable integers. The saddle point method* and III.6. *Integers free of small prime factors* in G. Tenenbaum's book [1], up to page 582.

Chapter III.5. Friable integers. The saddle point method

The following are Theorems 5.1 and 5.2 in [1]. We set $\Psi(x, y)$ to be the number of (natural numbers) $n \le x$ with the largest prime factor $\le y$. We define $(x \ge y \ge 2)$

$$u := \frac{\log x}{\log y} \text{ and } Z := \frac{\log x}{\log y}\log(1 + y/\log x) + \frac{y}{\log y}\log(1 + (\log x)/y)\,.$$

Theorem 1 *For any $x \ge y \ge 2$, $\Psi(x, y) \ll x\mathrm{e}^{-u/2}$.*

Theorem 2 *For any $x \ge y \ge 2$,*

$$\log(\Psi(x, y)) = Z\big(1 + O\big(1/\log y + 1/\log(\log(2x))\big)\big)$$

where the implicit constant is absolute.

The following are Theorem 5.3 and Corollary 5.4 (Ennola, 1969) in [1]. For $(a_j) \subset (0, +\infty)$ we denote by $N_k(z)$ the number of k-tuples $(\nu_1, \dots, \nu_k) \in \mathbb{N}_0^k$ such that $\nu_1 a_1 + \cdots + \nu_k a_k \le z$.

Theorem 3 *For any $k \in \mathbb{N}$ and $z \ge 0$,*

$$\frac{z^k}{k! \cdot a_1 \dots a_k} < N_k(z) \le \frac{(z + a_1 + \cdots + a_k)^k}{k! \cdot a_1 \dots a_k}\,.$$

Corollary 4 *For $2 \le y \le \sqrt{(\log x)\log(\log x)}$,*

$$\Psi(x,\, y) = \frac{1}{\pi(y)!}\prod_{p \le y}\frac{\log x}{\log p}\big(1 + O\big(y^2/(\log x)(\log y)\big)\big)$$

where the implicit constant is absolute.

The following are Theorem 5.5, Corollary 5.6 (Buchstab's identity) and Theorems 5.7 and 5.8 in [1].

Theorem 5 *For any* $x, y \geq 1$,

$$\Psi(x,\, y) = 1 + \sum_{p \leq y} \Psi(x/p,\, p)\,.$$

Corollary 6 *For any* $x \geq 1$ *and* $z \geq y \geq 1$,

$$\Psi(x,\, y) = \Psi(x,\, z) - \sum_{y < p \leq z} \Psi(x/p,\, p)\,.$$

Dickman's function $\rho(u)$ is defined by $\rho(u) = 1$ for $u \in [0, 1]$ and

$$\rho(u) = \rho(k) - \int_k^u \frac{\rho(v-1)}{v}\, \mathrm{d}v$$

for $u \in (k, k+1]$, $k \in \mathbb{N}$. It satisfies the delay DE $u\rho'(u) + \rho(u-1) = 0$ $(u > 1)$.

Theorem 7 *Dickman's function* $\rho(u)$ *has the following properties.*

1. *For* $u \geq 1$, $\rho(u) = \int_{u-1}^{u} \rho(v)\, \mathrm{d}v$.
2. *For* $u \geq 0$, $\rho(u) > 0$.
3. *For* $u > 1$, $\rho'(u) < 0$.
4. *For* $u \geq 0$, $\rho(u) \leq 1/\Gamma(u+1)$.

Theorem 8 *For* $x \geq y \geq 2$,

$$\Psi(x,\, y) = x\rho(u) + O(x/\log y)$$

where the implicit constant is absolute.

The following are Lemma 5.9, Theorem 5.10, Lemmas 5.11 and 5.12, Theorem 5.13 (de Bruijn; Alladi) and Corollaries 5.14 and 5.15 in [1]. For $s \in \mathbb{C}$ we set $I(s) = \int_0^s \frac{\mathrm{e}^t - 1}{t}\, \mathrm{d}t$.

Lemma 9 ...

Let

$$\widehat{\rho}(s) := \int_0^{+\infty} \mathrm{e}^{-st} \rho(t)\, \mathrm{d}t$$

be the Laplace transform of the Dickman function.

Theorem 10 *For* $s \in \mathbb{C}$, $\widehat{\rho}(s) = \exp(\gamma + I(-s))$.

For $u > 0$, $u \neq 1$, we let $\xi = \xi(u)$ be the unique real, non-zero root of $\mathrm{e}^{\xi} = 1 + u\xi$, and set $\xi(1) := 0$.

Lemma 11 ...

Lemma 12 ...

Theorem 13 *For $u \geq 1$,*

$$\rho(u) = \sqrt{\frac{\xi'(u)}{2\pi}} \mathrm{e}^{\gamma - u\xi + I(\xi)} (1 + O(1/u))\,.$$

Corollary 14 *For any $k \in \mathbb{N}_0$ and any real $u_0 > 1$ and $u \geq u_0$,*

$$\rho^{(k)}(u) = (-1)^k \xi(u)^k \rho(u)(1 + O(1/u))\,.$$

Corollary 15 *For $u \geq v \geq 0$,*

$$\rho(u - v) \ll \rho(u)\mathrm{e}^{v\xi(u)}$$

where the implicit constant is absolute.

The following are Lemma 5.16, Theorem 5.17 (Saias, 1989), Lemma 5.18, Corollary 5.19 (Hildebrand), Lemma 5.20 and Theorems 5.21 (Hildebrand–Tenenbaum), 5.22 and 5.23 in [1].

We denote by $\alpha = \alpha(x, y)$ the unique solution of

$$\frac{-\zeta'}{\zeta}(\alpha,\, y) = \sum_{p \leq y} \frac{\log p}{p^\alpha - 1} = \log x\,.$$

Lemma 16 ...

We denote by (H_ε) the domain $x > x_0(\varepsilon)$ and $\exp((\log(\log x))^{5/3+\varepsilon}) \leq y \leq x$ and set $L_\varepsilon(y) := \exp((\log y)^{3/5-\varepsilon})$.

Theorem 17 *Let $\varepsilon > 0$. For (x, y) in the domain (H_ε),*

$$\Psi(x,\, y) = \Lambda(x,\, y)\big(1 + O(1/L_\varepsilon(y))\big)$$

where $\Lambda(x, y)$ is the de Bruijn function (defined in (5.28)).

Lemma 18 ...

Corollary 19 *If $\varepsilon > 0$, $x \geq 3$ and $x \geq y \geq \exp\big((\log(\log x))^{5/3+\varepsilon}\big)$, then*

$$\Psi(x,\, y) = x\rho(u)\big(1 + O(\log(u+1)/\log y\big)$$

where the implied constant depends only on ε.

Lemma 20 ...

We set

$$\varphi_y(\sigma) := \sum_{p \leq y} \frac{\log p}{p^\sigma - 1} \quad \text{and} \quad \varphi_y'(\sigma) = \frac{\mathrm{d}\varphi_y(\sigma)}{\mathrm{d}\sigma}\,.$$

Theorem 21 *For $x \geq y \geq 2$,*

$$\Psi(x,\, y) = \frac{x^\alpha \zeta(\alpha,\, y)}{\alpha\sqrt{2\pi|\phi_y'(\alpha)|}}\bigl(1 + O(u^{-1} + y^{-1}\log y)\bigr)$$

where

$$|\phi_y'(\alpha)| = (1 + y^{-1}\log x)(\log x)(\log y)\bigl(1 + O(1/\log(u+1) + 1/\log y\bigr)$$

and the implicit constants are absolute. If $\varepsilon \in (0, \frac{1}{2})$ and $y \geq (\log x)^{1+\varepsilon}$ then

$$\Psi(x,\, y) = x\rho(u)\exp\bigl(O(\log(u+1)/\log y + u/L_\varepsilon(y))\bigr)$$

($\alpha = \alpha(x, y)$ is defined above).

("We shall not prove this result here.")

Theorem 22 *If $x \geq y \geq 2$, $c \geq 1$ and $t := (\log c)/\log y$, then*

$$\Psi(cx,\, y) = \Psi(x,\, y)c^{\alpha(x,y)}\bigl(1 + O((t^2+1)(u^{-1} + y^{-1}\log y)\bigr)$$

where the implicit constant is absolute.

Theorem 23 *If $x \geq y \geq 2$ and $c \geq 1$, then*

$$\Psi(cx,\, y) \leq c^\alpha \Psi(x,\, y)\bigl(1 + O(u^{-1} + y^{-1}\log y)\bigr)$$

where the implicit constant is absolute.

The following are Lemmas 5.24 and 5.25 and Theorem 5.26 (Rankin, 1938) in [1].

Lemma 24 ...

Lemma 25 ...

Next $j(n)$ is the Jacobsthal's function, the largest gap between two integers coprime to n, $P(Z) = \prod_{p \leq Z} p$ and $d_n = p_{n+1} - p_n$ (the gap between the n-th and the $(n+1)$-st prime).

Theorem 26 *For $Z \geq 100$,*

$$j(P(Z)) \gg \frac{Z \cdot (\log Z) \cdot \log(\log(\log Z))}{(\log(\log Z))^3}\,.$$

In particular, there is a $c > 0$ such that for infinitely many n,

$$d_n > \frac{c \cdot (\log p_n) \cdot (\log(\log p_n)) \cdot \log(\log(\log(\log p_n)))}{(\log(\log(\log p_n)))^2}\,.$$

Chapter III.6. Integers free of small prime factors

Now $\Phi(x, y)$ is the number of (natural numbers) $n \le x$ with the minimum prime factor $> y$. The following are Lemma 6.1 and Theorem 6.2 in [1]. We set $\zeta(1, y) = \prod_{p \le y} \frac{1}{1-1/p}$.

Lemma 27 ...

Theorem 28 *If $x \ge y \ge 2$ then*

$$\Phi(x,\, y) = \frac{x}{\zeta(1,\, y)} + O(\Psi(x,\, y))$$

where the implicit constant is absolute.

The following are Theorems 6.3 and 6.4, Corollary 6.5 and Theorem 6.6 in [1].

Theorem 29 *For $x, y \ge 1$,*

$$\Phi(x,\, y) = 1 + \sum_{y<p\le x} \sum_{\nu \ge 1} \Phi(x/p^{\nu},\, p)\,.$$

The function $\omega(u)$ is the unique continuous solution for $u > 1$ to the delay DE

$$(u\omega(u))' = \omega(u-1) \quad (u > 2)\,,$$

with the initial condition $u\omega(u) = 1$ for $u \in [1, 2]$.

Theorem 30 *For $x \ge y \ge 2$,*

$$\Phi(x,\, y) = \frac{x\omega(u) - y}{\log y} + O\big(x/(\log y)^2\big)$$

where the implicit constant is absolute.

Corollary 31 *For $u \ge 1$,*

$$\omega(u) = \mathrm{e}^{-\gamma} + O\big(u^{-u/2}\big)\,.$$

Theorem 32 *For $u \in \mathbb{R}$ one has that $|\omega'(u)| \le \rho(u)$ and for $u \ge 1$ it holds that $\omega(u) = \mathrm{e}^{-\gamma} + O\big(\rho(u)/\log(u+1)\big)$.*

The following are Theorems 6.7 and 6.8 and Corollary 6.9 in [1]. Now

$$\widehat{\omega}(s) := \int_0^{+\infty} \mathrm{e}^{-su}\omega(u)\,\mathrm{d}u$$

is the Laplace transform of $\omega(u)$.

Theorem 33 *The function $\widehat{\omega}(s)$ extends to $\mathbb{C}\setminus\{0\}$ to a meromorphic function*

$$1+\widehat{\omega}(s)=\frac{1}{s\widehat{\rho}(s)}\,.$$

When $s\notin(-\infty,0)$ then $1+\widehat{\omega}(s)=\mathrm{e}^{J(s)}$ ($J(s)=\int_0^{+\infty}\mathrm{e}^{-s-t}\,\mathrm{d}t/(s+t)$).

Let $H(u)=\exp(u/\log^2(u+2))$ $(u\geq 0)$.

Theorem 34 *For an absolute constant $a>0$ and $u\geq 0$ we have that $\omega(u)-\mathrm{e}^{-\gamma}\ll\rho(u)H(u)^{-a}$ and the same bound holds for $\omega'(u)$.*

Corollary 35 *For $j\in\mathbb{N}$ and $u\geq 0$,*

$$\omega^{(j)}(u)\ll\rho^{(j)}(u)H(u)^{-a}\,.$$

The following are Theorem 6.10, Lemmas 6.11–13, Corollaries 6.14–6.16 and Theorem 6.17 in [1]. Let

$$Y_\varepsilon(y):=\exp((\log y)^{3/2-\varepsilon}),\ E(x,\,y):=H(u)^{-c}L_\varepsilon(y)^{-1}+Y_\varepsilon^{-1}.$$

Also,

$$\mu_y(u):=\int_0^{+\infty}\omega(u-v)y^{-v}\,\mathrm{d}v,\ W(x,\,y):=x\mu_y(u)\frac{\mathrm{e}^y\log y}{\zeta(1,\,y)}\,.$$

Theorem 36 *For any $\varepsilon>0$ and $x\geq y\geq 2$ we have $\Phi(x,y)-W(x,y)\ll\Psi(x,y)E(x,y)$ in H_ε and $\ll\Psi(x,y)$ outside H_ε.*

Lemma 37 ...

Lemma 38 ...

Lemma 39 ...

Corollary 40 *For $\varepsilon>0$ and $(x,y)\in H_\varepsilon$,*

$$\Phi(x,\,y)-\frac{x}{\zeta(1,\,y)}\ll\frac{x\rho(u)}{\log y}\big(H(u)^{-c_5}+Y_\varepsilon^{-1}\big)\,.$$

Corollary 41 *For $\varepsilon>0$ and $(x,y)\in H_\varepsilon$,*

$$\Phi(x,\,y)=(x\omega(u)-y)\frac{\mathrm{e}^\gamma}{\zeta(1,\,y)}+O\big(x\rho(u)(H(u)^{-c_6}+Y_\varepsilon^{-1})/(\log y)^2\big)\,.$$

Corollary 42 *For $x\geq 2y\geq 5$,*

$$\Phi(x,\,y)=\frac{\mathrm{e}^\gamma(x\omega(u)-y)}{\zeta(1,\,y)}\big(1+O(\mathrm{e}^{-u/3}/\log y)\big)$$

where the implicit constant is absolute.

Theorem 43 *Let $n \in \mathbb{N}$ and $y \in \mathbb{R}$ with $n \geq y \geq 2$. Then for any $\varepsilon > 0$,*

$$K(n,\, y) \ll_\varepsilon \rho(u) 2^{(1+\varepsilon)u} + n^{-1+\varepsilon}$$

where $u := \log n / \log y$. For any $\varepsilon > 0$, the approximation

$$\begin{aligned} K(n,\, y) &= \Xi(u)\big(1 + O\big((\log(u+1) + 2^{-u}\log(\log y))/\log y\big)\big) \\ &+ O\big(\rho(u/2)^2/\exp((\log y)^{3/2-\varepsilon})\big) \end{aligned}$$

holds uniformly in the domain $n \geq 3$, $\exp(\log(\log n)^{5/3+\varepsilon}) \leq y \leq n$.

Here $K(n,y)$ is so called Kubilius gauge and

$$\Xi(u) = \frac{1}{2}\int_{\mathbb{R}} |\omega(v) - \mathrm{e}^{-\gamma}|\rho(u-v)\,\mathrm{d}v + \rho(u)/2\,.$$

References

[1] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, AMS, Providence, RI 2015 (Third Edition. Translated by Patrick D. F. Fion)