Lecture 12. Distribution of additive functions and mean values of multiplicative functions

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In the twelfth lecture we cover Chapter III.4. *Distribution of additive functions and mean values of multiplicative functions* in G. Tenenbaum's book [6], up to page 511.

Chapter III.4. Distribution of additive functions and mean values of multiplicative functions

The following are Theorems 4.1 (Erdős–Wintner, 1939) and 4.2 (Delange), Lemma 4.3, Theorems 4.4 (Delange), 4.5 (Halász), 4.6 (Wirsing) and 4.7, Lemmas 4.8, 4.9 (Gallagher), 4.10 (Montgomery–Wirsing) and 4.11, Corollary 4.12, Lemma 4.13 and Theorems 4.14 (Hall and Tenenbaum, 1991) and 4.15 (Erdős and Kac, 1939; Rényi and Turán, 1958) in [6].

Theorem 1 *An additive function $f\colon \mathbb{N}\to\mathbb{R}$ has a limiting distribution $\iff$ there is an $R>0$ such that the three series* (a) $\sum_{|f(p)|>R}\frac{1}{p}$, (b) $\sum_{|f(p)|\le R}\frac{f(p)^2}{p}$ *and* (c) $\sum_{|f(p)|\le R}\frac{f(p)}{p}$ *simultaneously converge. If it is the case then the characteristic function of the limit law is given for any $\tau\in\mathbb{R}$ by the convergent product*

$$\varphi(\tau)=\prod_p(1-1/p)\sum_{\nu\ge 0}\exp(i\tau f(p^\nu))/p^\nu\,.$$

The limit law is pure, and it is continuous iff $\sum_{f(p)\ne 0}\frac{1}{p}=+\infty$.

This theorem is from [2].

In the following $D=\{z\in\mathbb{C}:\ |z|\le 1\}$ is the unit complex disc.

Theorem 2 *Let $g\colon\mathbb{N}\to D$ be multiplicative.* (i) *If the mean value $M(g):=$ $\lim_{x\to+\infty}\frac{1}{x}\sum_{n\le x}f(n)\ne 0$ then* (a) *the series $\sum_p(1-g(p))/p$ converges and* (b) *$\exists\,\nu\in\mathbb{N}$ such that $g(2^\nu)\ne -1$.* (ii) *If* (a) *holds then the mean value $M(g)$ exists and is given by the formula*

$$M(g)=\prod_p\big(1-1/p\big)\sum_{\nu\ge 0}g(p^\nu)/p^\nu\,.$$

Lemma 3 *Let $H > 0$ and let $(u_n), (v_n) \subset \mathbb{C}$ be such that always $1+u_n+v_n \neq 0$ and $\sum_n(|u_n|^2+|v_n|) \leq H$. Then $\prod_n(1+u_n+v_n)$ converges iff $\sum_n u_n$ converges. Then*

$$\left|\prod_n(1+u_n+v_n)\right| \leq \exp\left(6H + \sum_n \mathrm{Re}(u_n)\right).$$

Theorem 4 *Let $g\colon \mathbb{N} \to D$ be multiplicative. If $\sum_p(1-\mathrm{Re}(g(p)))/p < +\infty$ then for $x \to +\infty$ it holds that*

$$\frac{1}{x}\sum_{n\leq x} g(n) = \prod_{p\leq x}\left(1-1/p\right)\sum_{\nu\geq 0} g(p^\nu)/p^\nu + o(1).$$

"Theorem 4.4 has not been published by Delange but has been object of several oral expositions."

Theorem 5 *Let $g\colon \mathbb{N} \to D$ be multiplicative. If there is a $\tau \in \mathbb{R}$ such that* (s): $\sum_p(1-\mathrm{Re}(g(p)p^{-i\tau}))/p$ *converges, then for $x \to +\infty$ it holds that*

$$\frac{1}{x}\sum_{n\leq x} g(n) = \frac{x^{i\tau}}{1+i\tau}\prod_{p\leq x}\left(1-1/p\right)\sum_{\nu\geq 0}\frac{g(p^\nu)}{p^{\nu(1+i\tau)}} + o(1).$$

If the series (s) *does not converge for any τ then*

$$\frac{1}{x}\sum_{n\leq x} g(n) = o(1).$$

This theorem is from [3].

Theorem 6 *Let $g\colon \mathbb{N} \to [-1,]$ be multiplicative. Then*

$$\lim_{x\to+\infty}\frac{1}{x}\sum_{n\leq x} g(n) = \prod_p\left(1-1/p\right)\sum_{\nu\geq 0} g(p^\nu)/p^\nu$$

"where the infinite product is to be taken as zero when it diverges."

This theorem is from [7].

Let $G(x) := \sum_{n\leq x} g(n)$, $F(s) := \sum_n g(n)/n^s$ $(\sigma > 1)$ and for $T, \alpha > 0$, $H_T(\alpha)^2 := \sum_{\substack{k\in\mathbb{Z}\\ |k|\leq T}} \frac{1}{k^2+1} \max_{\substack{\sigma=1+\alpha\\ |\tau-\alpha|\leq 1/2}} |F(s)|^2$.

Theorem 7 *For any multiplicative $g\colon \mathbb{N} \to D$, with this notation it uniformly holds for $T > 0$ and $x \geq 2$ that*

$$G(x) \ll \frac{x}{\log x}\int_{1/\log x}^{1} H_T(\alpha)\mathrm{d}\alpha/\alpha + \frac{x}{T}.$$

Lemma 8 *Let $M \subset \mathbb{R}$ be compact and $f_n\colon M \to \mathbb{R}$, $n \in \mathbb{N}$, be continuous functions such that $f_1 \leq f_2 \leq \ldots$ and for any $x \in M$ one has that $\lim_{n\to\infty} f_n(x) = +\infty$. Then this limit is uniform,*

$$\forall\, c\, \exists\, n_0\, \forall\, x \in M\, \forall\, n \geq n_0 : \ f_n(x) > c.$$

Proof. In not, there is a c and sequences $n_1 < n_2 < \dots$ and $(x_m) \subset M$ such that for any m one has that $f_{n_m}(x_m) \leq c$. We may assume that $\lim_{m\to\infty} x_m = x_0 \in M$. Let n be arbitrary but fixed. Then for any large m,

$$f_{n_m}(x_m) \geq f_n(x_m) \geq f_n(x_0) - 1$$

because $\lim_{m\to\infty} f_n(x_m) = f_n(x_0)$. Thus $f_n(x_0) \leq c + 1$ and it is not true that $\lim_{n\to\infty} f_n(x_0) = +\infty$. $\square$

In the case of convergence of f_n to a function f this is Dini's theorem.

The next three lemmas are needed for the proof of Theorem 7.

Lemma 9 *Let $\lambda_1, \dots, \lambda_N$ for $N \in \mathbb{N}$ be distinct real numbers and let $\delta_n := \min_{m\neq n} |\lambda_m - \lambda_n|$. Then for all $T > 0$ and $a_1, \dots, a_N \in \mathbb{C}$*

$$\int_{-T}^{T} \Big| \sum_{n=1}^{N} a_n \mathrm{e}(\lambda_n t) \Big|^2 \,\mathrm{d}t \ll \sum_{n=1}^{N} |a_n|^2 (T + 1/\delta_n)\,,$$

where the implied constant is absolute. In particular, for any Dirichlet series $\sum_{n\geq 1} a_n/n^s$ with abscissa of convergence $< \alpha$, it uniformly holds for $T > 0$ and $\sigma \geq \alpha$ that

$$\int_{-T}^{T} |F(s)|^2 \,\mathrm{d}\tau \ll \sum_{n\geq 1} \frac{|a_n|^2}{n^{2\sigma}} (T + n)\,.$$

Lemma 10 *If Dirichlet series $A(s) := \sum_{n\geq 1} a_n/n^s$ and $B(s) := \sum_{n\geq 1} b_n/n^s$ converge for $\sigma \geq 1$ and $|a_n| \leq b_n$, then for $T \geq 0$ and $\sigma > 1$,*

$$\int_{-T}^{T} |A(s)|^2 \,\mathrm{d}\tau \leq 3 \int_{-T}^{T} |B(s)|^2 \,\mathrm{d}\tau\,.$$

Lemma 11 *Let $g\colon \mathbb{N} \to D$ be multiplicative. Then for $\sigma > 1$ we have that $\sum_{n\geq 1} g(n)/n^s = (1 + D(s))F_1(s)J(s)$ where*

$$D(s) = \sum_{\nu\geq 1} \frac{g(2^\nu)}{2^{\nu s}}, \;\; F_1(s) = \exp\big(\textstyle\sum_{p>2} g(p)/p^s\big)$$

and where $J(s)$ is a function holomorphic on $\sigma > \frac{1}{2}$ that for $\sigma \geq 1$ satisfies $1 \ll J(s) \ll 1$ and $J'(s) \ll 1$.

The following is an effective form of Halász Theorem 5.

Corollary 12 *Let $g\colon \mathbb{N} \to D$ be multiplicative and for $x, T \geq 2$ let*

$$m(x, T) := \min_{|\tau|\leq T} \sum_{p\leq x} \frac{1 - \mathrm{Re}(g(p)p^{-i\tau})}{p} \;\; \textit{and} \;\; R(x, T) := \frac{1 + m(x, T)}{\mathrm{e}^{m(x,T)}} + \frac{1}{T}\,.$$

Then $\sum_{n\leq x} g(n) \ll xR(x, T)$.

Lemma 13 *Let* $h\colon \mathbb{R}\to\mathbb{R}$ *be a* 2π*-periodic function that has on* $[0,2\pi]$ *bounded variation and has* $\overline{h} := \frac{1}{2\pi}\int_0^{2\pi} h(t)\,\mathrm{d}t$*. Let* $M(h) := \sup_t |h(t)|$ *and* $V(h) := \int_0^{2\pi} |\mathrm{d}h(t)|$*. Then for any* $\tau, w, z \in \mathbb{R}$ *with* $\tau \neq 0$ *and* $1 < w < z$*,*

$$\sum_{w<p\le z} \frac{h(\tau \log p)}{p} = \overline{h}\log\big(\frac{\log z}{\log w}\big) + O\big(\frac{V(h)}{|\tau|\log w} + \frac{M(h)+(1+|\tau|)V(h)}{\mathrm{e}^{\sqrt{\log w}}}\big)\,.$$

Theorem 14 *Let* $\varphi_0 \in (0,2\pi)$ *be the unique solution of* $\sin\varphi + (\pi-\varphi)\cos\varphi = \pi/2$ *and* $K := \cos\varphi_0 \approx 0.32867$*. Then for any* $x \ge 2$ *and any multiplicative function* $g\colon \mathbb{N}\to[-1,1]$*,*

$$\sum_{n\le x} g(n) \ll x\exp\big(-K\sum_{p\le x}\frac{1-g(p)}{p}\big)$$

where the implicit constant does not depend on g*.*

This theorem is from [4].

Let

$$\Phi(y) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} \mathrm{e}^{-t^2/2}\,\mathrm{d}t$$

be the normal distribution function. As usual, $\omega(n)$ is the number of prime factors of n.

Theorem 15 *For any* $N \in \mathbb{N}\setminus\{1\}$ *and* $y\in\mathbb{R}$*,*

$$N^{-1}|\{n \le N:\ \omega(n) \le \log\log N + y\sqrt{\log\log N}\}| = \Phi(y) + O(1/\sqrt{\log\log N})$$

where the implicit constant is absolute.

This theorem is from [1] and [5].

References

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