# **Lecture 11.** Normal order

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In the eleventh lecture we cover Chapter III.3. *Normal order* in G. Tenenbaum's book [4], up to page 475.

Two (arithmetic) functions $f, g\colon \mathbb{N} \to \mathbb{C}$ are *normal orders* of one another if $f(n) = (1 + o(1))g(n)$ for $n \in X$, $n \to \infty$, where $X \subset \mathbb{N}$ has natural density 1. We write $f(n) = (1 + o(1))g(n)$ (a. e.).

### Chapter III.3. Normal order

The following are Theorems 3.1 (Turán–Kubilius inequality), 3.2–3.5, Corollary 3.6 and Theorems 3.7–3.10 in [4].

We have to introduce some notation. Let $f\colon \mathbb{N} \to \mathbb{C}$. For $N \in \mathbb{N}$ we define $Z_N = Z_{f,N} = \sum_{p \le N} \zeta_p$ where $\zeta_p = \zeta_p(f)$ are independent random variables on an abstract prob. space $S = (\Omega, P)$ with laws $P(\zeta_p = f(p^\nu)) = (1 - 1/p)p^{-\nu}$, $\nu \in \mathbb{N}_0$ ($f(1) = 0$). We have the *empirical variance* $\mathbb{V}_N(f) = N^{-1}\sum_{n \le N} |f(n) - \mathbb{E}_N(f)|^2$ and the *semi-empirical variance* $\mathbb{V}^*_N(f) = N^{-1}\sum_{n \le N} |f(n) - \mathbb{E}(Z_{f,N})|^2$. We set

$$B_f(N)^2 = \sum_{p \le N} \mathbb{E}(|\zeta_p|^2) = \sum_{p^\nu \le N} \frac{|f(p^\nu)|^2}{p^\nu}\big(1 - 1/p\big)\,.$$

Also,

$$C_N := \sup_{f \ne 0} \frac{\mathbb{V}_N(f)}{\mathbb{V}(Z_{f,N})} \ \text{ and } \ C^*_N := \sup_{f \ne 0} \frac{\mathbb{V}^*_N(f)}{\mathbb{V}(Z_{f,N})}\,.$$

Finally,

$$\varepsilon_N := \frac{8}{N}\Big(\sum_{\substack{p^\nu q^\mu \le N \\ p \ne q}} p^\nu q^\mu\Big)^{1/2} + \frac{4}{N}\Big(2\sum_{p^\nu \le N} p^\nu \sum_{p \le N} \frac{1}{p}\Big)^{1/2}$$

**Theorem 1** *If $f\colon \mathbb{N} \to \mathbb{C}$ is additive then for any $N \in \mathbb{N}$,*

$$\mathbb{V}^*_N(f) \le (4 + \varepsilon_N)B_f(N)^2 \le (8 + 2\varepsilon_N)\mathbb{V}(Z_{f,N})\,.$$

*In particular,* $\max(C_N, C^*_N) \le 8 + O\big(\sqrt{\log\log N/\log N}\big)$ *($N \ge 3$).*

[4] gives cryptically some references for this result: Turán [6] and (added by us) Kubilius [3]. *Paul (Pál) Turán (1910–1976)* was a Hungarian mathematician and *Jonas Kubilius (1921-2011)* was a Lithuanian mathematician who was rector of Vilnius University for 32 years.

**Theorem 2** *For any $a_1, \ldots, a_N \in \mathbb{C}$,*

$$\sum_{p^\nu \le N} \frac{p^\nu}{1-1/p} \Big| \sum_{\substack{n \le N \\ p^\nu \,\|\, n}} a_n - \frac{1-1/p}{p^\nu} \sum_{n \le N} a_n \Big|^2 \le N C_N^* \sum_{n \le N} |a_n|^2 \, .$$

**Theorem 3** *If $f\colon \mathbb{N} \to \mathbb{C}$ is additive and if $B_f(N) = o(\mathbb{E}(Z_{f,N}))$ $(N \to \infty)$, then $g(n) := \mathbb{E}(Z_{f,N})$ is a normal oder for $f$.*

**Theorem 4** *For any function $\xi(n) \to \infty$ and any $N \ge 3$,*

$$|\{n \le N : \ |\omega(n) - \log\log N| > \xi(N)\sqrt{\log\log N}\}| < N/\xi(N)^2 \, .$$

*The same holds for $\Omega(n)$.*

This is classical Turán's form [5] of the classical 1917 theorem of Hardy and Ramanujan [2].

**Theorem 5** *Let $f\colon \mathbb{N} \to [0, +\infty)$ be multiplicative. If for some constants $A, B > 0$ it holds that* (i) *for any $y \ge 0$ one has that $\sum_{p \le y} f(p) \log p \le Ay$ and* (ii) *$\sum_p \sum_{\nu \ge 2} f(p^\nu) \log(p^\nu)/p^\nu \le B$, then for any $x > 1$,*

$$\sum_{n \le x} f(n) = (A + B + 1) \frac{x}{\log x} \sum_{n \le x} \frac{f(n)}{n} \, .$$

By [4] this theorem is from [1].

**Corollary 6** *Let $\lambda_1 > 0$ and $\lambda_2 \in [0, 2)$. Suppose that $f\colon \mathbb{N} \to \mathbb{R}$ is multiplicative and satisfies $0 \le f(p^\nu) \le \lambda_1 \lambda_2^{\nu-1}$ for any $p$ and $\nu \in \mathbb{N}$. Then for any $x \ge 1$,*

$$\sum_{n \le x} f(n) \ll x \prod_{p \le x} \big(1 - 1/p\big) \sum_{\nu \ge 0} \frac{f(p^\nu)}{p^\nu} \, ,$$

*with an implicit constant that is independent of $f$ and $\le 4(1 + 9\lambda_1 + \lambda_1\lambda_2/(2 - \lambda_2)^2)$.*

For $t > 0$ we set

$$\omega(n, t) = \sum_{p \,|\, n,\, p \le t} 1 \ \text{ and } \ \Omega(n, t) = \sum_{p^\nu \,|\, n,\, p \le t} \nu \, .$$

**Theorem 7** *Let $y_0 > 0$. Then for any $y \in [0, y_0]$ and $x \ge t \ge 2$ it uniformly holds that*

$$\sum_{n \le x} y^{\omega(n,\, t)} \ll x (\log t)^{y-1} \, .$$

*If in addition $y_0 < 2$ then the same holds for $\Omega(n, t)$.*

**Theorem 8** *For any $x \geq t \geq 3$ and $0 \leq \xi \leq \sqrt{\log\log t}$ it uniformly holds that*

$$|\{n \leq x :\ |\omega(n,\, t) - \log\log t| > \xi\sqrt{\log\log t}\}| \ll x\mathrm{e}^{-\xi^2/3}\,.$$

*For $\xi \ll (\log\log t)^{1/6}$, $\cdot/3 \rightsquigarrow \cdot/2$. The same holds for $\Omega(n,t)$ if $0 \leq \xi \leq c\sqrt{\log\log t}$ for some constant $c \in (0,1)$.*

**Theorem 9** *Let $\varepsilon > 0$ and $\xi(n) \to \infty$. Then*

$$\sup_{\xi(n)\leq t\leq n} \Big| \frac{\omega(n,\, t) - \log\log t}{\sqrt{2\log\log t \cdot \log\log\log t}} \Big| \quad (a.\ e.)\,.$$

By $p_j(n)$ we denote the $j$-th prime factor of $n$.

**Theorem 10** *Let $\varepsilon > 0$ and $\xi(n) \to \infty$. Then*

$$\sup_{\xi(n)\leq j\leq n} \Big| \frac{\log\log p_j(n) - j}{\sqrt{2j \cdot \log j}} \Big| \leq 1 + \varepsilon \quad (a.\ e.)\,.$$

# References


[1] R. R. Hall and G. Tenenbaum, *Divisors*, CUP, Cambridge 1988

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[3] J. Kubilius, *Veroyatostnye metody v teorii chisel*, Gos. Izd. Polit. i Nauch. Lit. Litovskoj SSR, Vil'nyus 1962 (2nd ed., *Probabilistic Methods in Number Theory*, originally *Tikimybiniai Metodai Skaičių Teorijoje*)

[4] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, AMS, Providence, RI 2015 (Third Edition. Translated by Patrick D. F. Fion)

[5] P. Turán, On a theorem of Hardy and Ramanujan, *J. London Math. Soc.* **9** (1934), 274–276

[6] P. Turán, Über einige Verallgemeinerungen eines Satzes von Hardy und Ramanujan, *J. London Math. Soc.* **11** (1936), 125–133