Grounded String Representations of Series-Parallel Graphs without Transitive Edges

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[6] Universität Würzburg

EuroCG 2026

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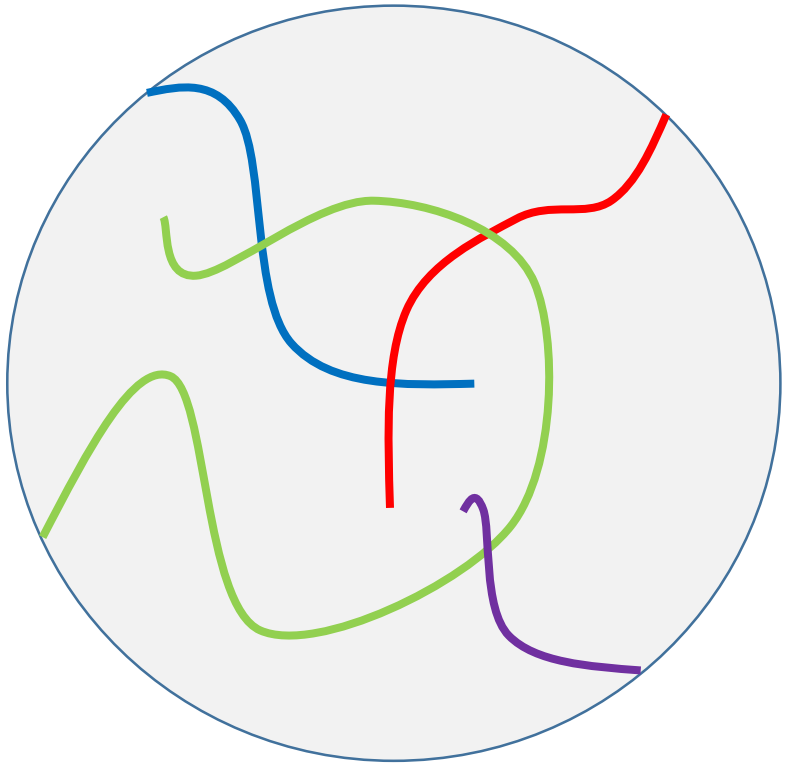
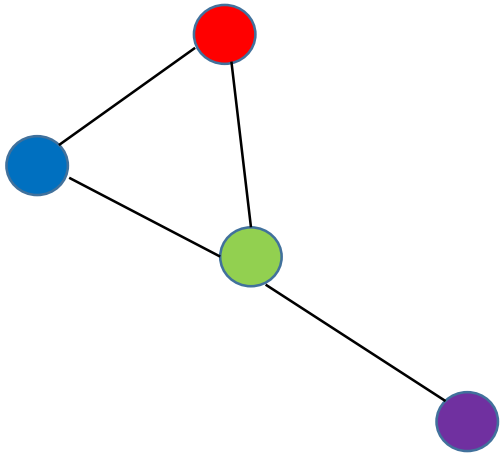
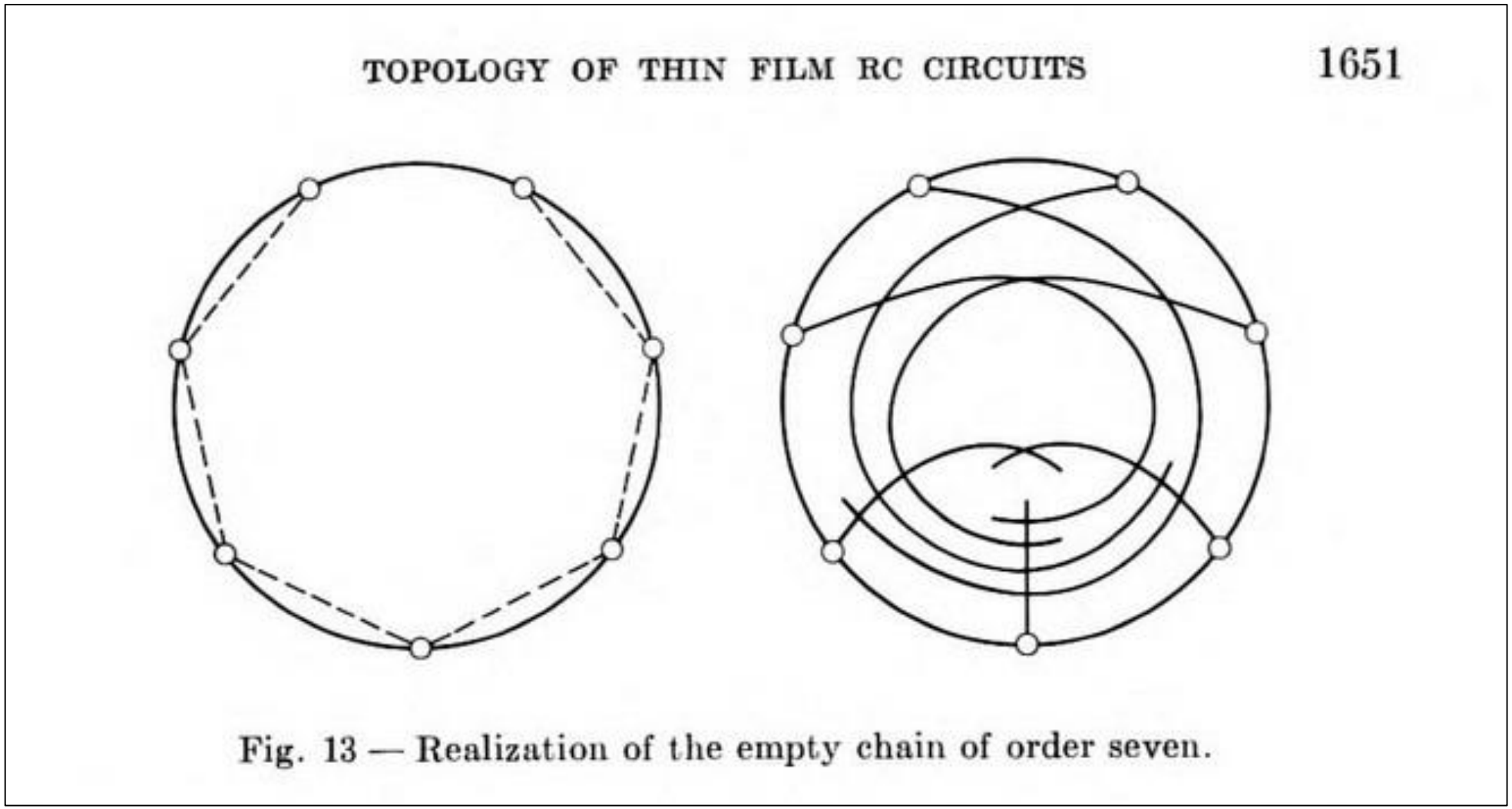
Outerstring graphs





Outerstring graphs

Implicitly introduced by Sinden (1966)



TOPOLOGY OF THIN FILM RC CIRCUITS 1651

Fig. 13 — Realization of the empty chain of order seven.

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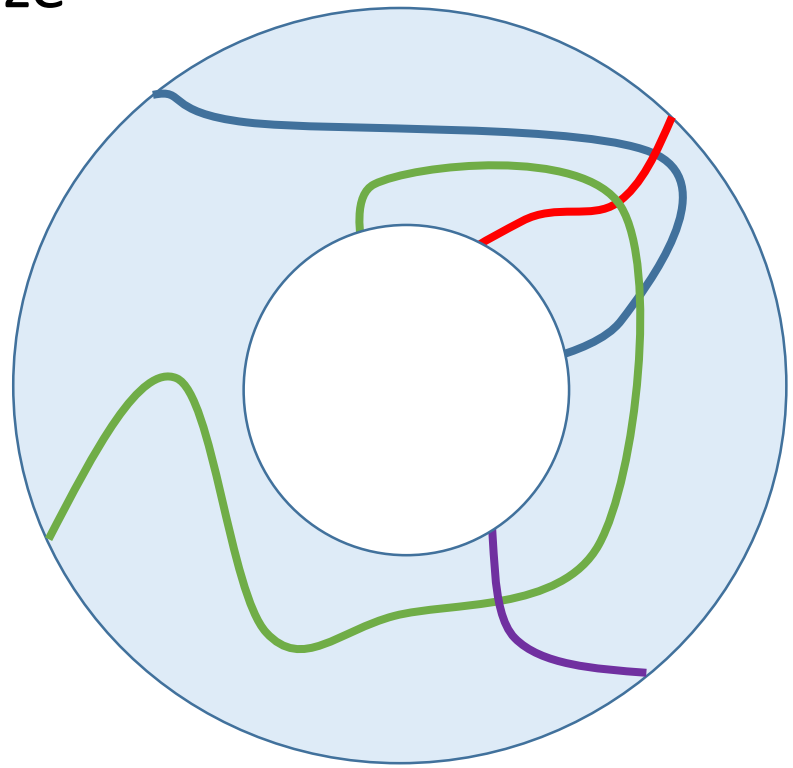

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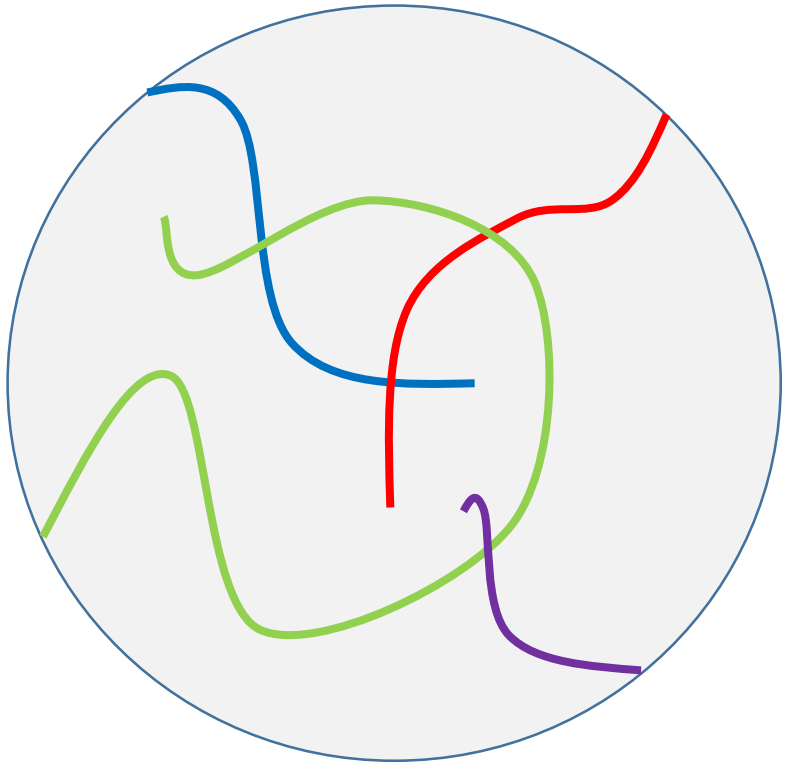
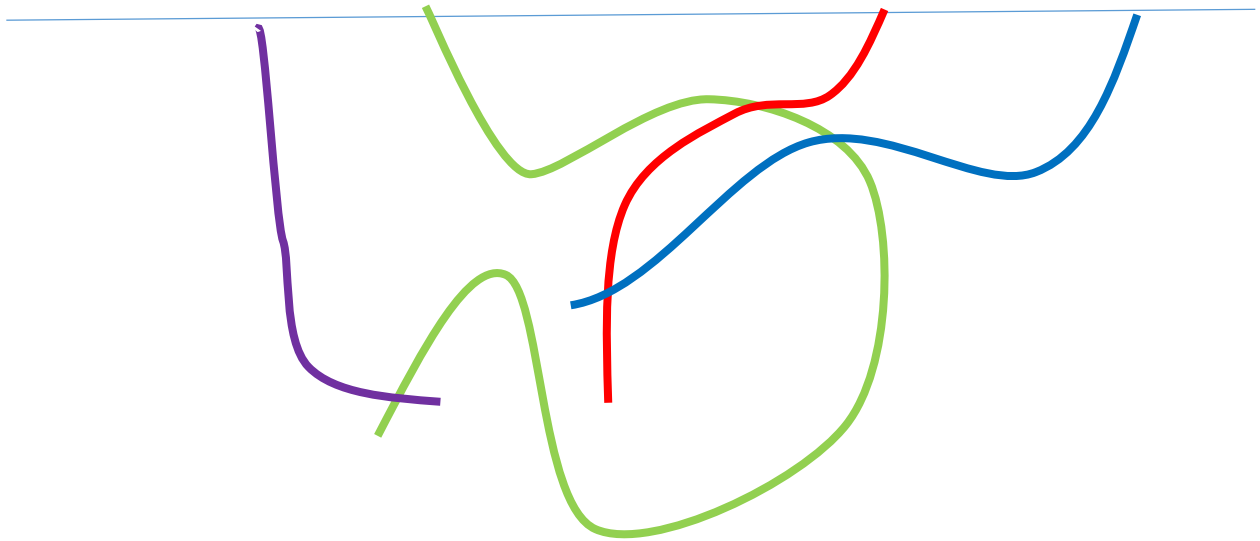
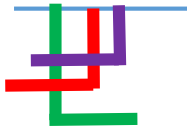
Rok, Walczak (2019): Outer-string graphs are χ-bounded

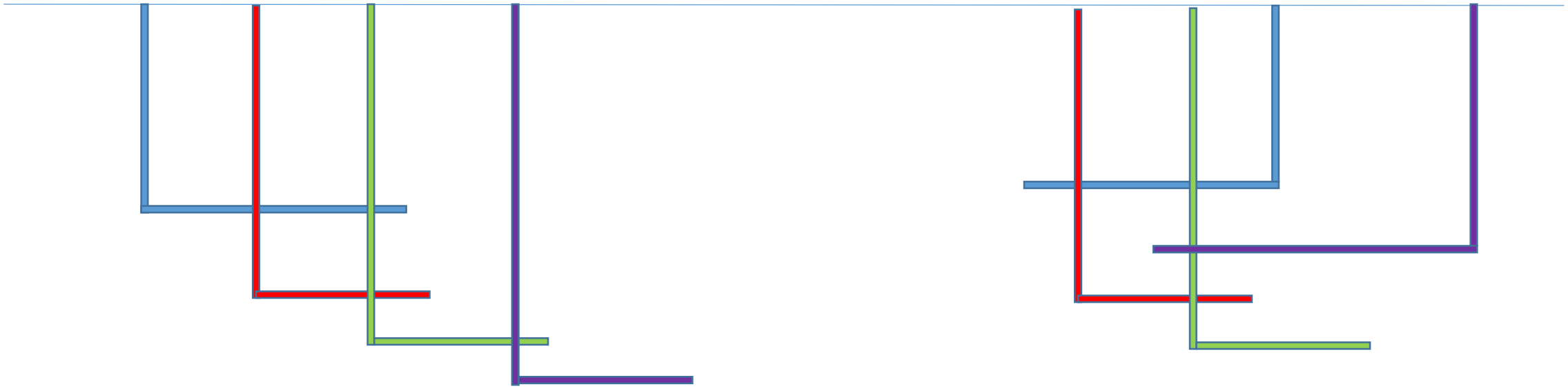
Outerstring graphs = Grounded string graphs





Grounded L-graphs and Grounded L-⅃-graphs





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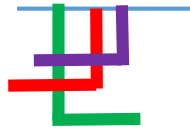

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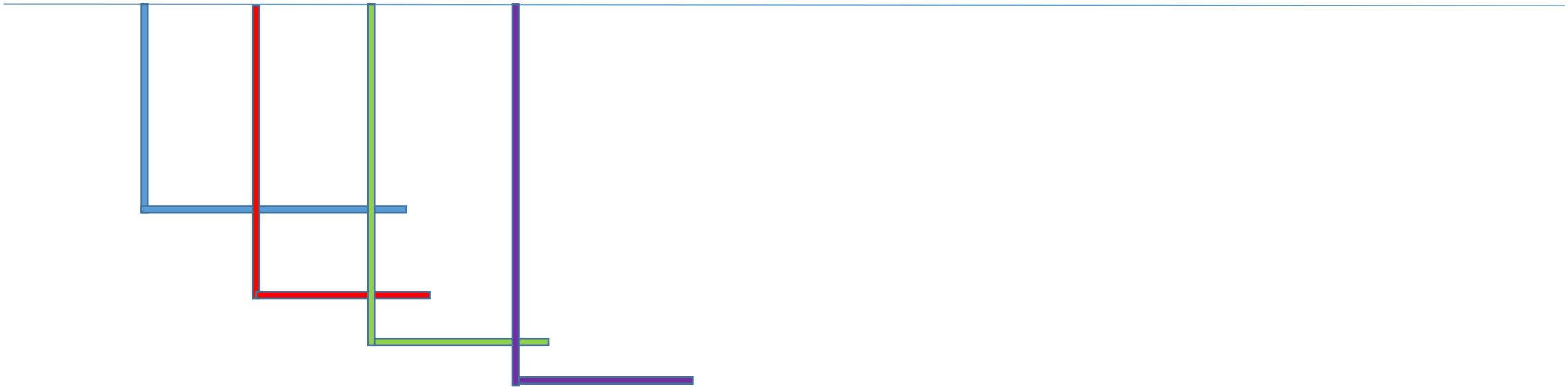

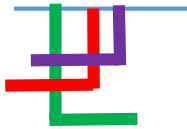

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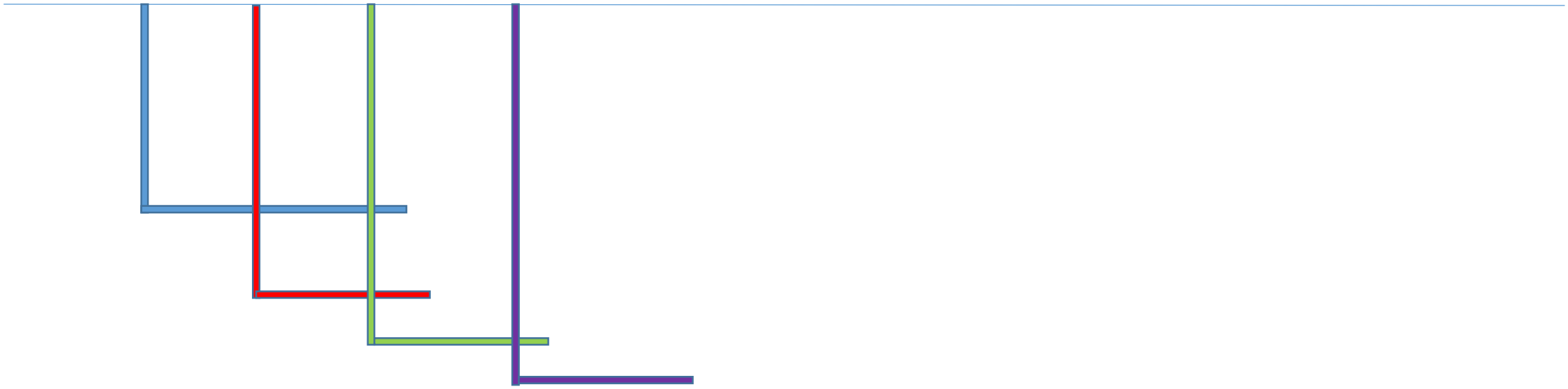
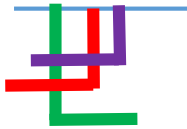




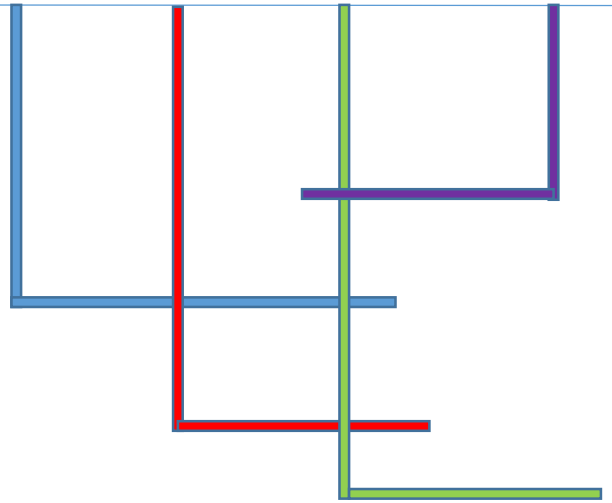
Grounded L-graphs and Grounded L-⅃-graphs

Our results:

1) Necessary condition for a biconnected graph to be outerstring

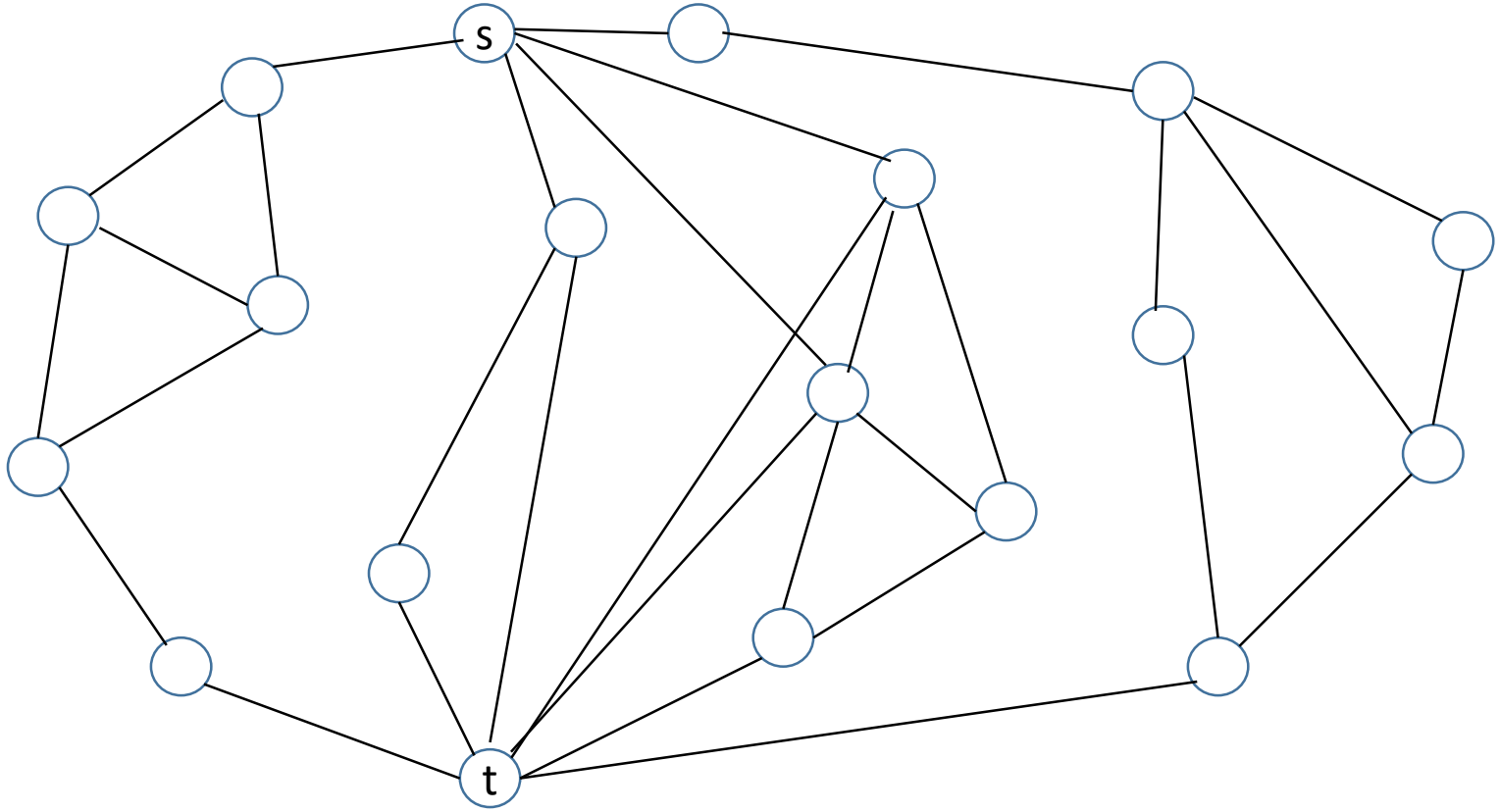
2) Necessary and sufficient condition for a biconnected series-parallel graph without transitive edges to be a grounded L-⅃-graph

3) Testing if a biconnected series-parallel graph without transitive edges admits a grounded L-⅃-representation in linear time



Biconnected Outerstring Graphs

Separation pair *s,t*



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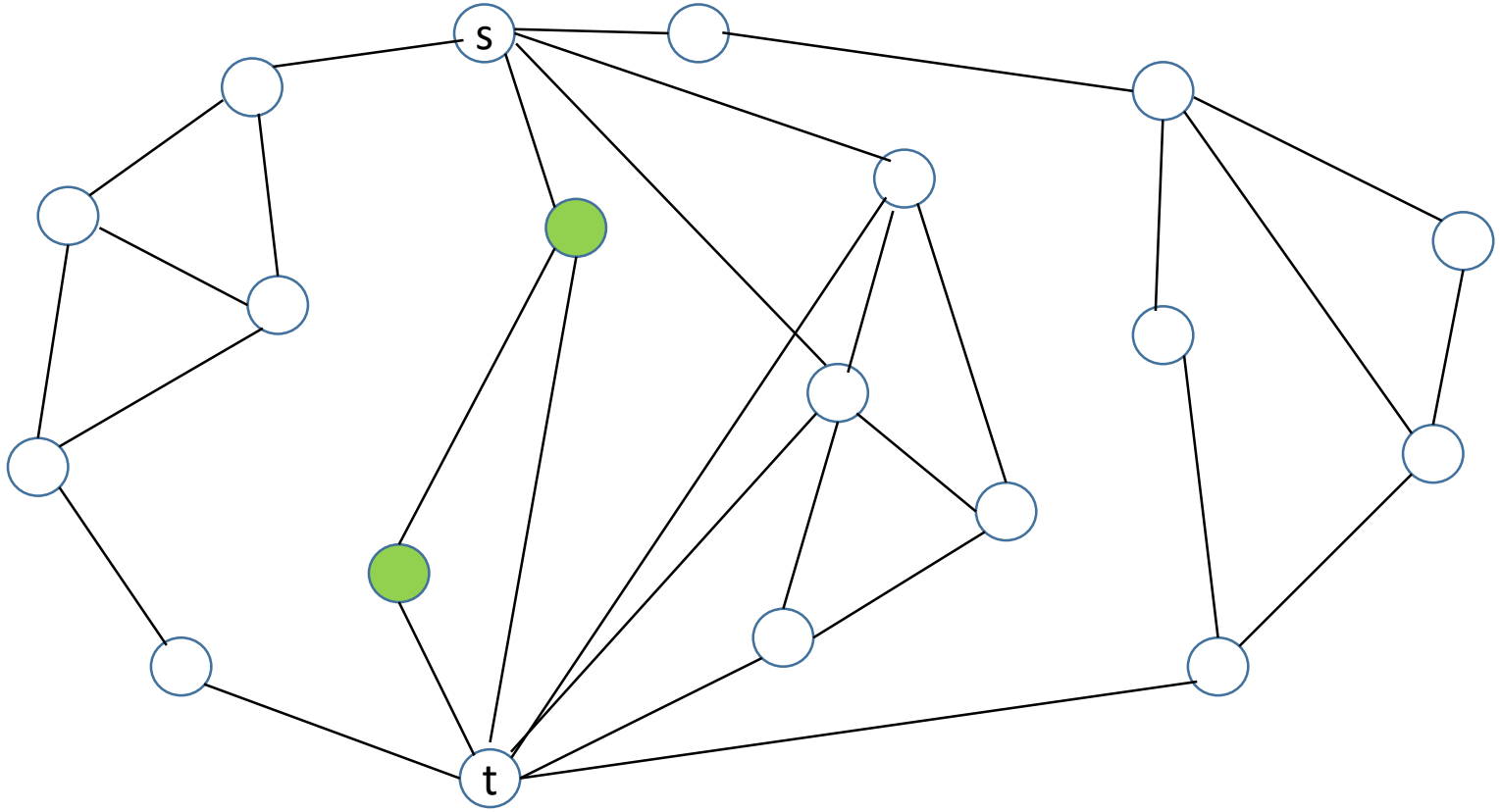
light component – every vertex is adjacent to *s* or to *t* (or both)



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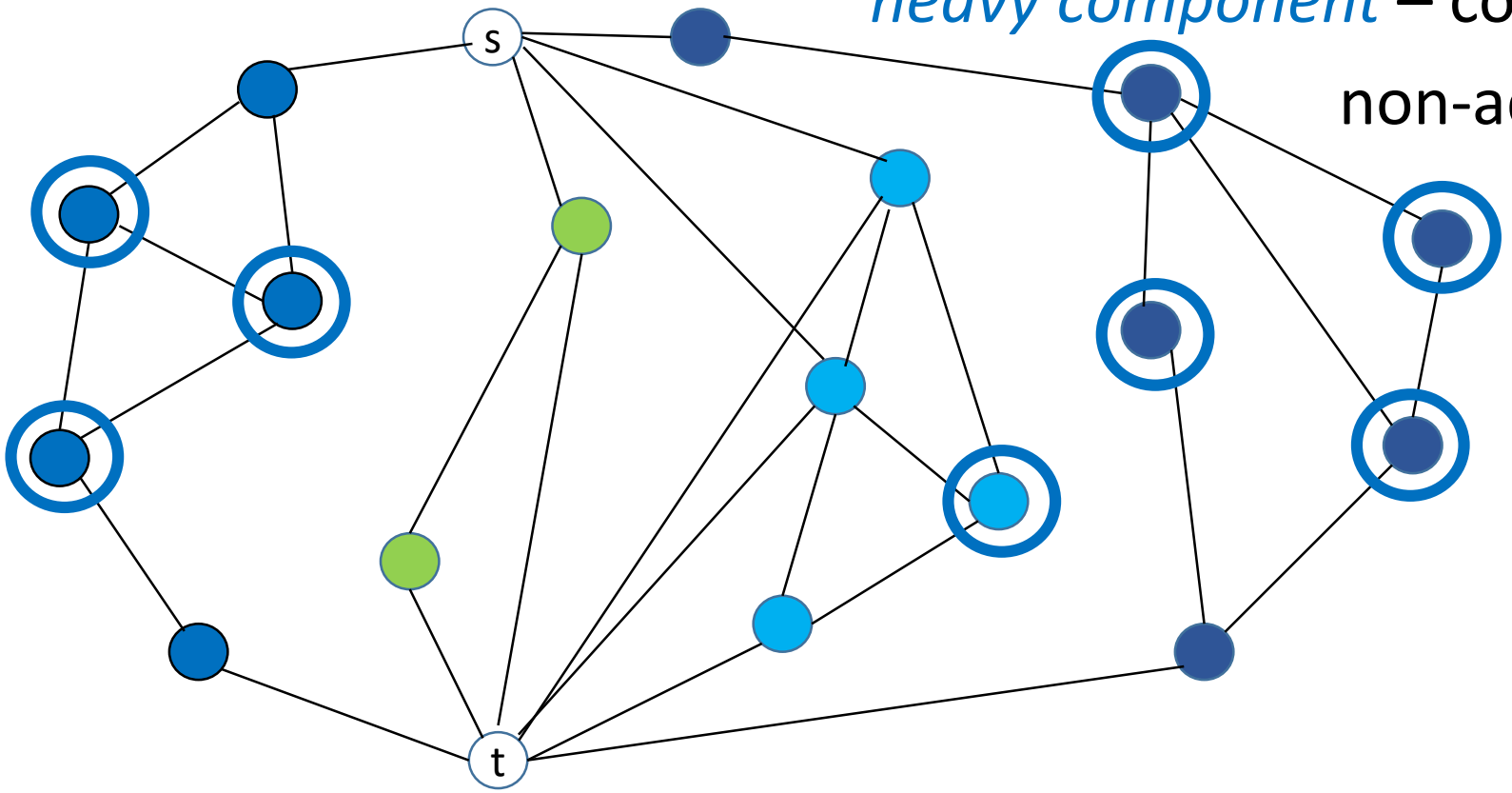

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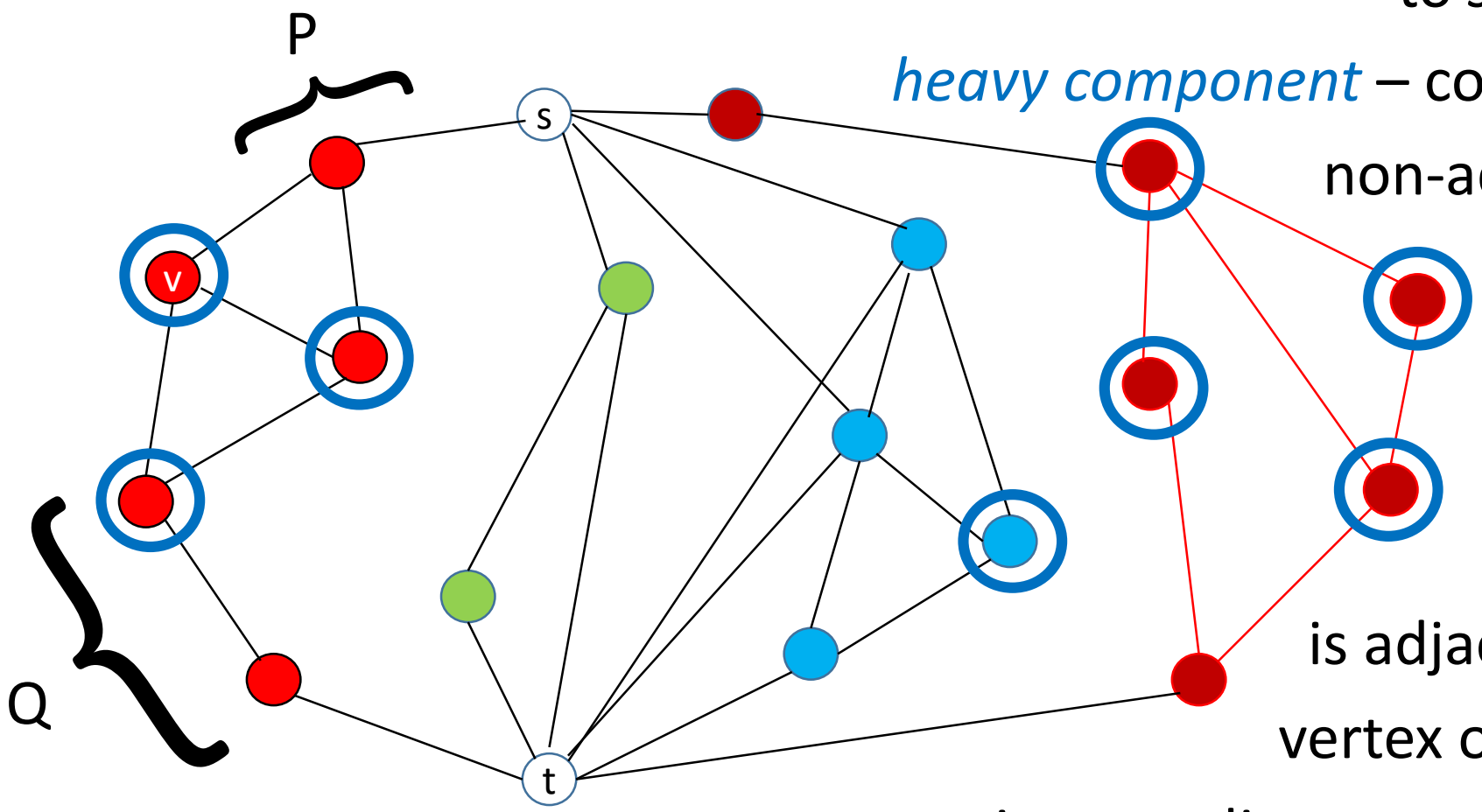

light component – every vertex is adjacent to *s* or to *t* (or both)

heavy component – contains a vertex non-adjacent to *s* and *t*

critical component – contains a path *s* – *P* – *v* – *Q* – *t* s.t. no internal vertex of *P* is adjacent to *t*; no internal vertex of *Q* is adjacent to *s*; and *v* is non-adjacent to *s* and *t* (*v* is heavy)

Biconnected Outerstring Graphs

Lemma: Let *s,t* be a separation pair in a biconnected outerstring graph *G*. Then 1) if *s,t* are non-adjacent, *G* has at most 2 critical components w.r.t. *s,t*;

2) if *s,t* are adjacent and there is an outerstring representation with just 1 crossing point of *s* and *t*, then *G* has at most 4 critical components w.r.t. *s,t*.

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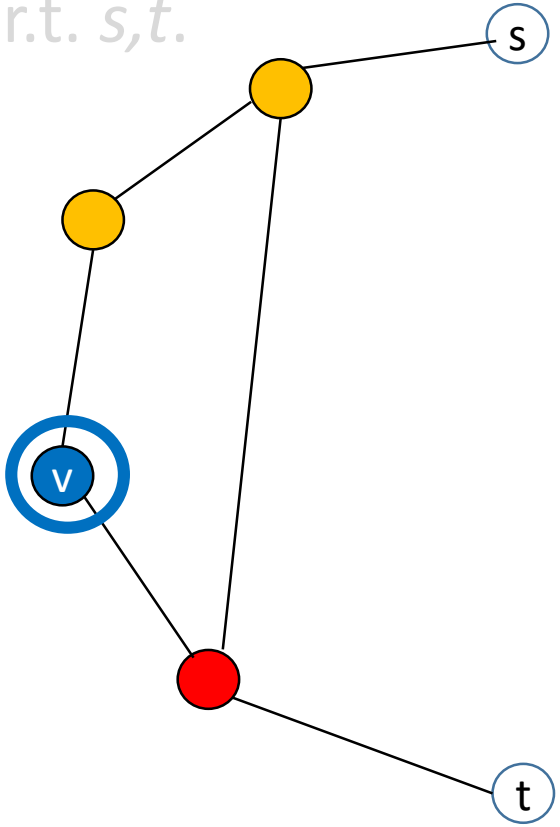

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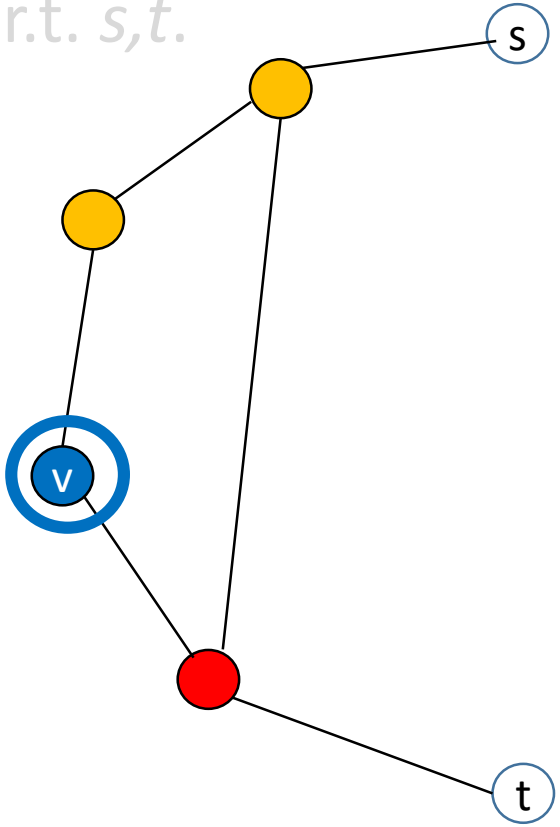

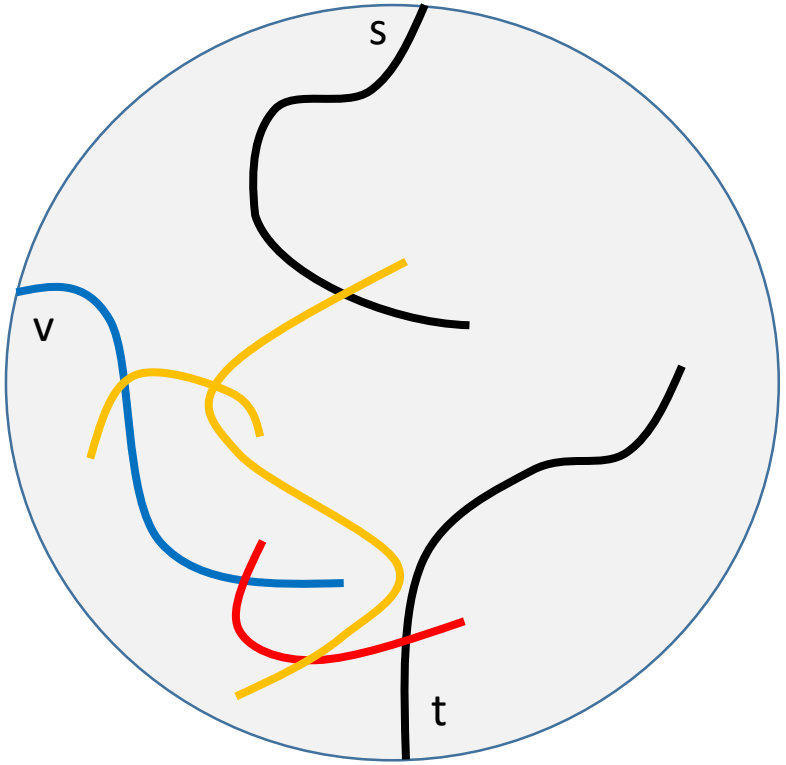

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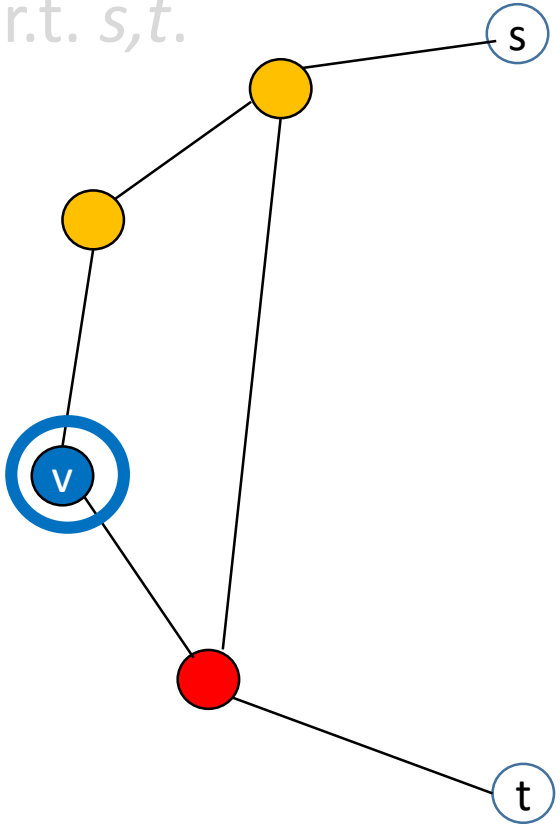

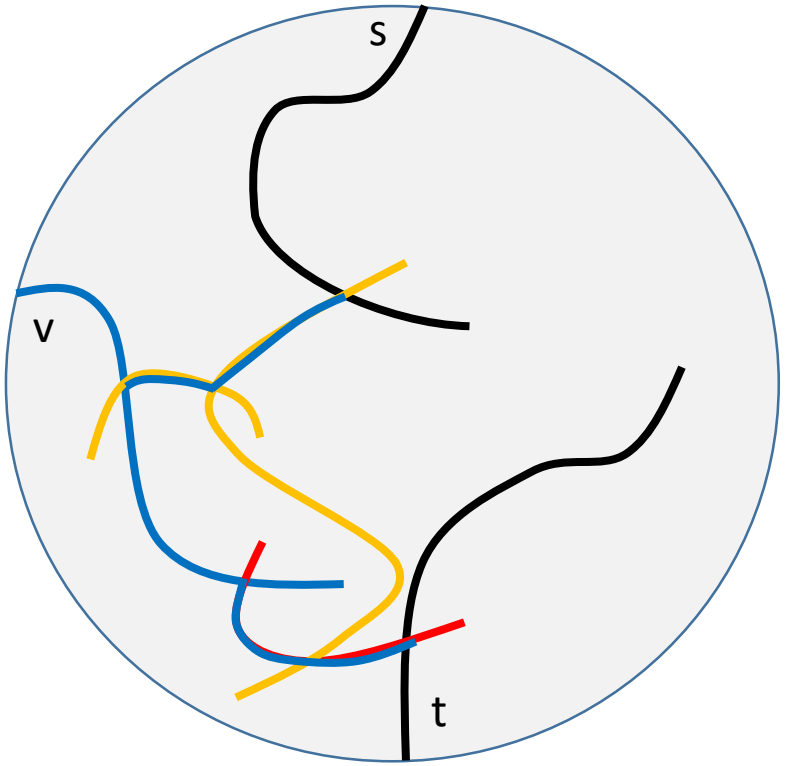

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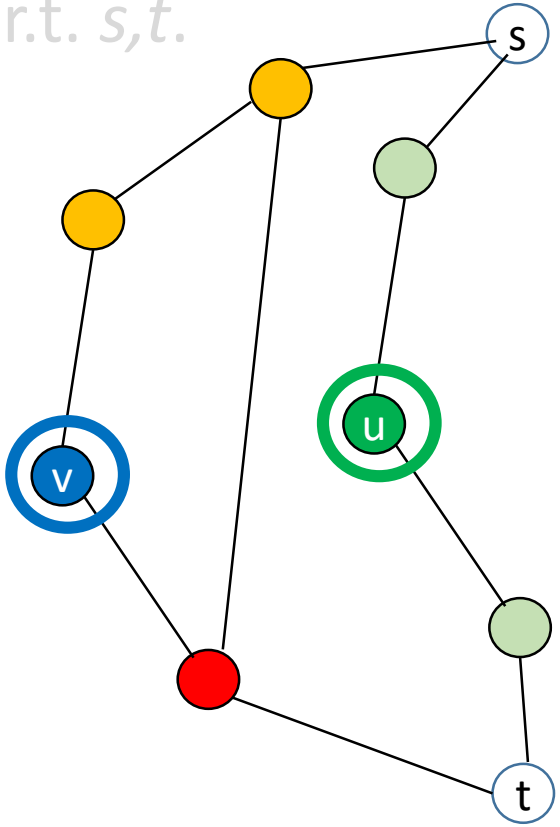

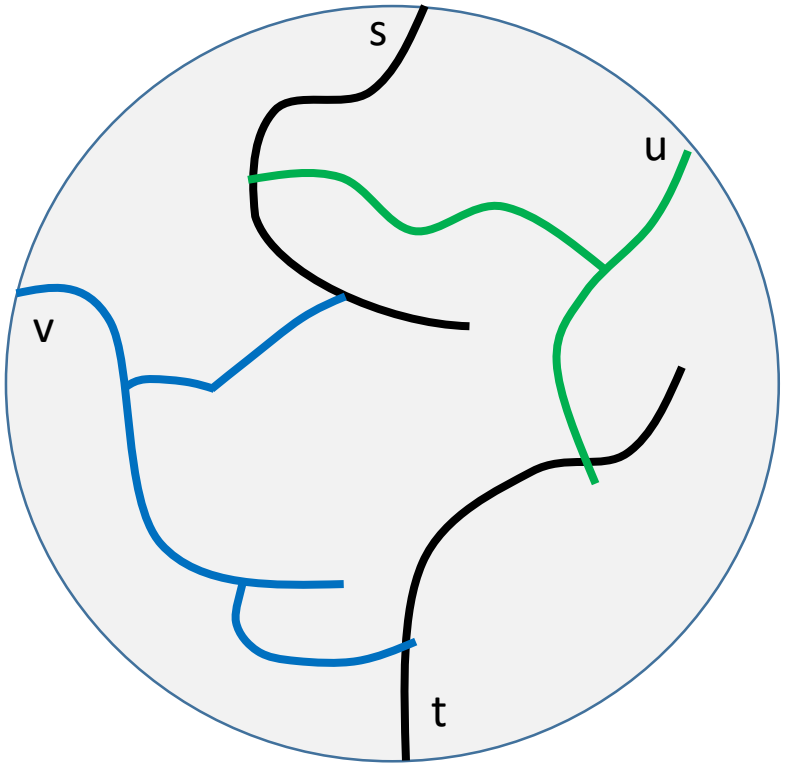

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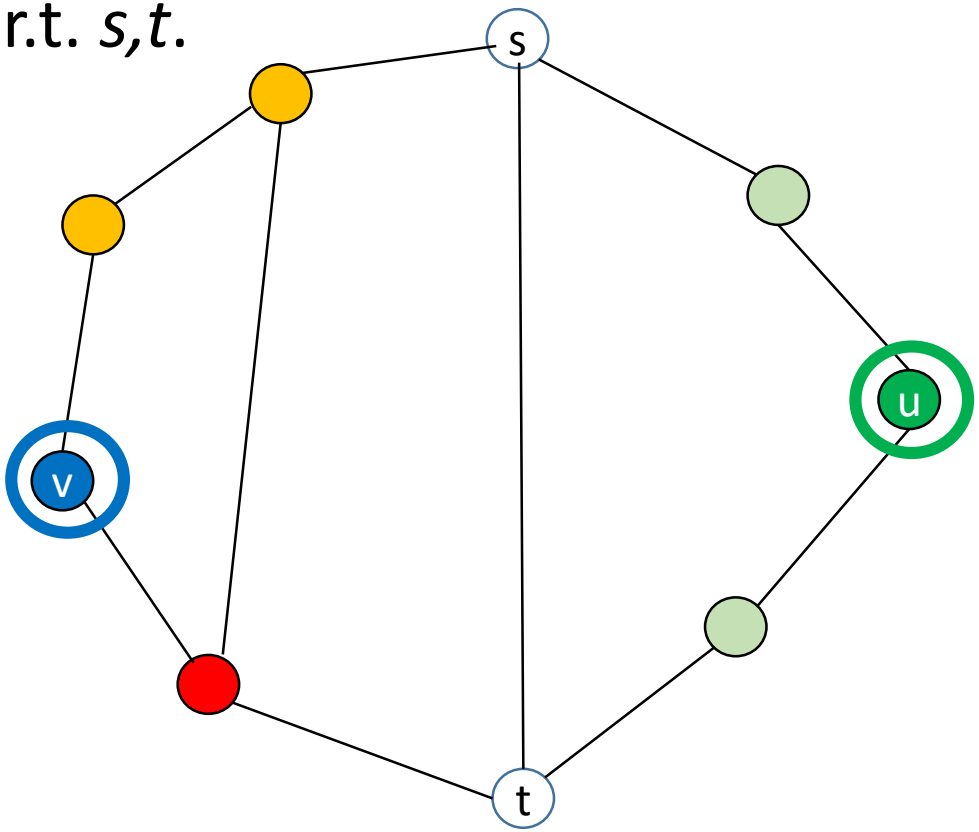

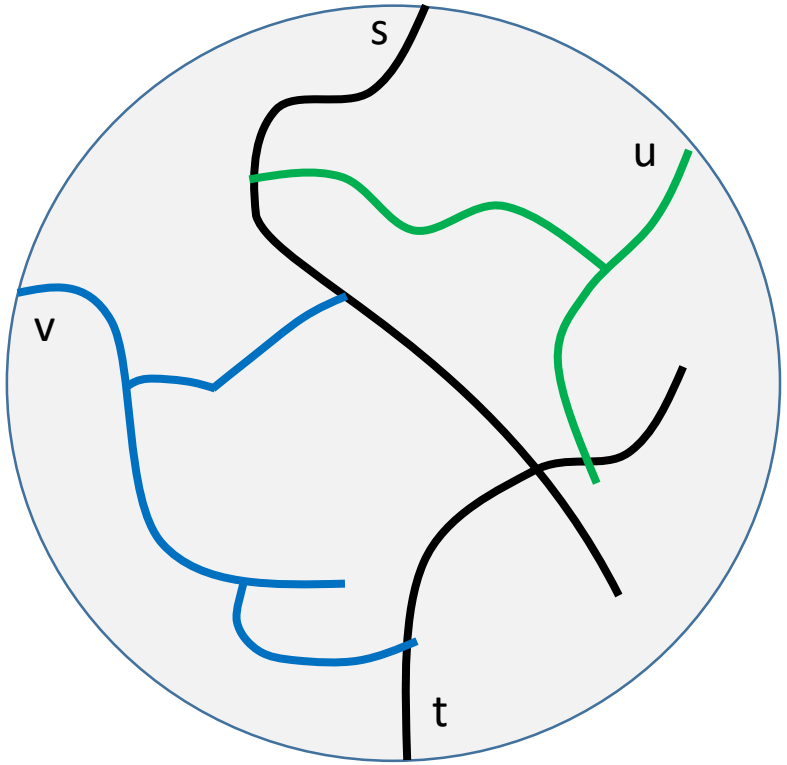

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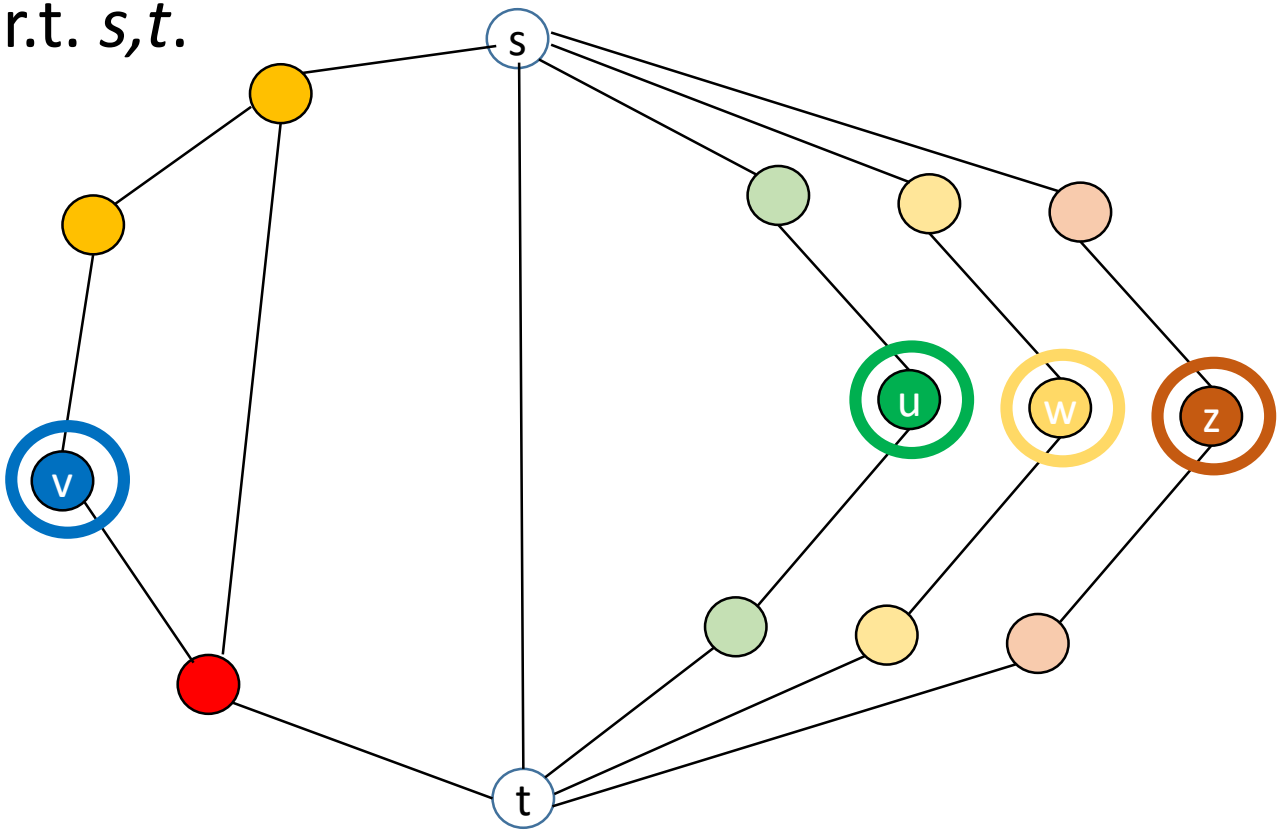

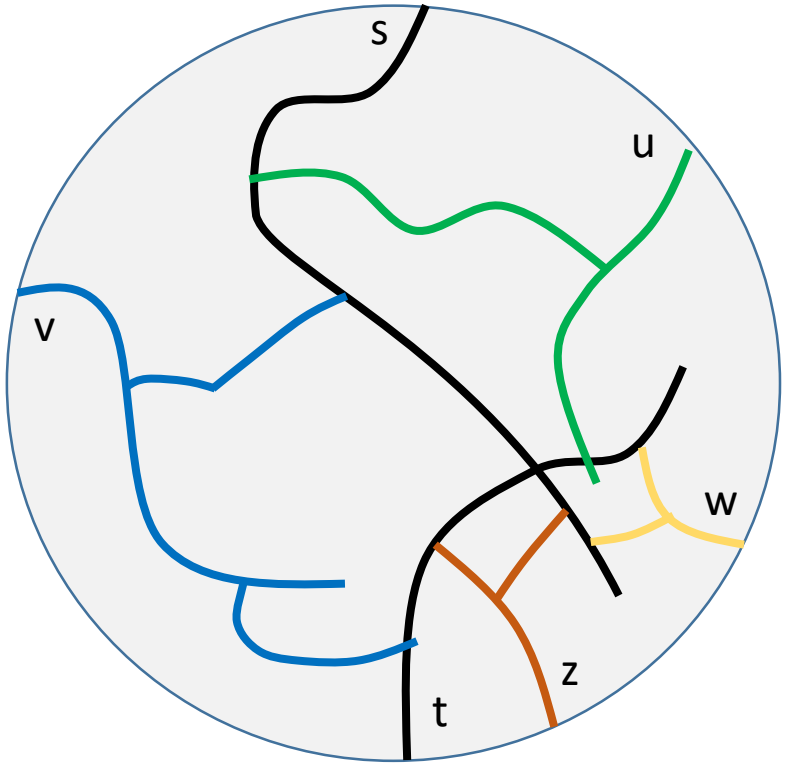

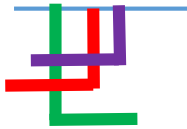

Biconnected Series-Parallel Grounded L-⅃-graphs

Theorem: Let *G* be a biconnected series-parallel graph with no transitive edges. Then the following are equivalent:

1) For every separation pair *s,t, G* has at most 2 heavy components w.r.t. *s,t*;

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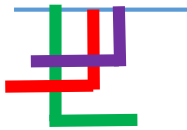

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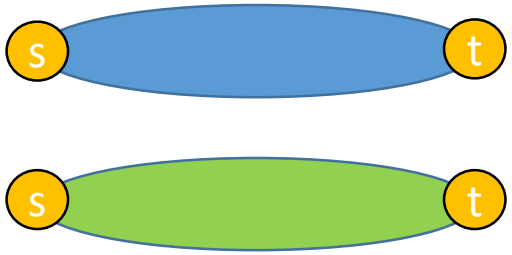

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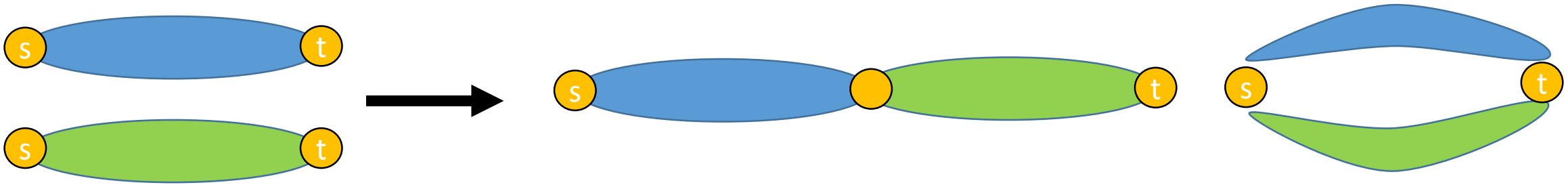

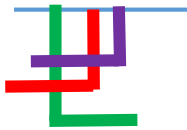

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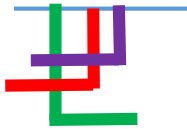
Biconnected series-parallel = Biconnected graphs without a subdivided K_4



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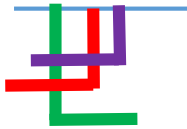
Observation: For biconnected series-parallel graphs without transitive edges, heavy = critical



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Observation: For biconnected series-parallel graphs without transitive edges, heavy = critical

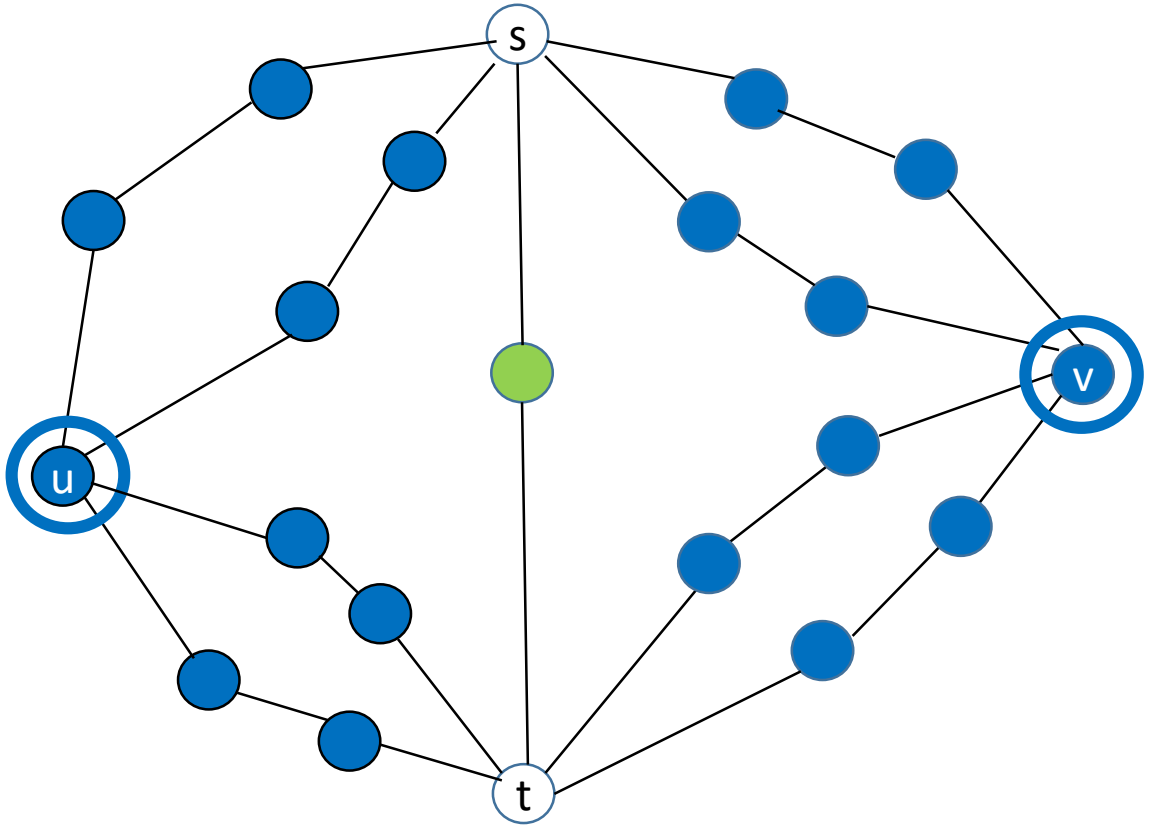
Proof of the theorem: 3) $\Rightarrow$ 2) is obvious

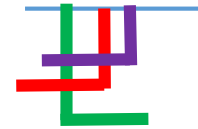
2) $\Rightarrow$ 1) follows from the Lemma and the Observation

1) $\Rightarrow$ 3) an ad hoc construction

Biconnected Series-Parallel Grounded L-graphs

Example: This graph is biconnected series-parallel, has no transitive edges and for every separation pair *s,t, it* has at most 2 heavy components w.r.t. *s,t*. But it has no grounded **L**-representation.





Open Problems

Problem 1: Is recognition of grounded **L-**⌋**-**graphs NP-hard?

Problem 2: Characterize biconnected series-parallel graphs (with transitive edges permitted) that admit grounded **L-**⌋**-**representations. Can these graphs be recognized in polynomial time?