Discrete Math (Solutions) Tutorial 3
Kisel/Tiwary

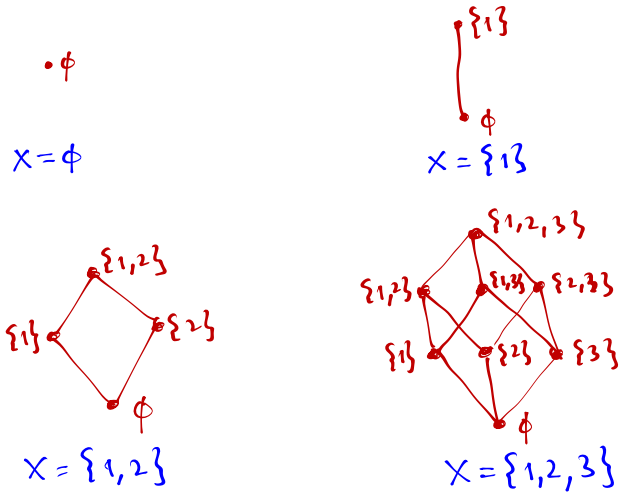
Problem 1. *Give an example of a partial order on X where X has 15 elements such that width of your partial order is 3 and length is 5.*

Solution. There are many solutions but we give one example. Take our underlying set to be $\{2, 4, 8, 16, 32, 3, 9, 27, 81, 243, 5, 25, 125, 625, 3125\}$. And as for our ordering let's say that $x \leq y$ if x divides y. This is clearly a partial ordering and our longest chain is 5, for example $2 \leq 4 \leq 8 \leq 16 \leq 32$. And to see that our biggest antichain has size 3 take for example $2, 3, 5$, this is clearly an antichain. To see that this a largest antichain, let $S_1 = \{2, 4, 8, 16, 32\}, S_2 = \{3, 9, 27, 81, 243\}$, and $S_3 = \{5, 25, 125, 625, 3125\}$. Any antichain can contain at most one element from each of the S_i's and therefore has size at most three. □

Problem 2. *Let X be a finite set.*

1. *Prove that $(2^X, \subseteq)$ is a poset.*
2. *Draw the Hasse diagram for this poset when X is $\emptyset, [1], [2], [3]$*

Solution. For any $A \in 2^X$ we have $A \subseteq A$. So, the relation is reflexive.
Let $A, B \in 2^X$. If $A \subseteq B$ and $B \subseteq A$, then $A = B$. So, the relation is anti-symmetric.
Let $A, B, C \in 2^X$. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$. So, the relation is transitive.
Since the relation $\subseteq$ is reflexive, anti-symmetric, and transitive over 2^X it is an ordering and $(2^X, \subseteq)$ is a poset.
Below are the Hasse diagram for this poset when X is $\emptyset, [1], [2], [3]$.



□

Problem 3. *Let $p \in \mathbb{N}, p \geqslant 2$. Define the relation R over N as follows: xRy iff p divides $x - y$.*

1. *Prove that R is an equivalence relation.*
2. *Identify the equivalence classes of R.*

Solution.

1. Since p divides $x - x$ for all $x \in \mathbb{N}, R$ is reflexive.

   If p divides $x - y$ then it divides $y - x$ so R is symmetric.

   Suppose $x, y), (y, z) \in R$. That is, p divides $x - y$ and $y - z$. So There exist integers k, l such that $x - y = kp, y - z = lp$. Adding these two together we get that $x - z = (x - y) + (y - z) = (k + l)p$. So p divides $x - z$. That is $(x, z) \in R$. Thus the relation is transitive as well.

   Since R is reflexive, symmetric, and transitive over $\mathbb{N}$, it is an equivalence relation over $\mathbb{N}$.

2. For any natural number i the equivalence class $[i]_R$ is the set of all natural number that leave the same remainder as i when dividing by p. Any natural number leaves a remainder of $0, 1, \ldots,$ or $p - 1$ when dividing by p. So there are p equivalences classes each corresponding to one of the remainders. The class corresponding to remainder $i \in \{0, 1, \ldots, p-1$ contains all natural numbers of the form $kp+i$ where k is a natural number.

□

Problem 4. *Prove that a minimum element, if it exists, is unique.*

Solution. Let $P = (X, \preccurlyeq)$ be a poset and let $a, b \in X$ be minimum elements. Since a is minimum, we have $a \preccurlyeq b$. Since b is minimum we have $b \preccurlyeq a$. Therefore by anti-symmetry we have $a = b$. Thus two distinct minimum elements cannot exist. □

Problem 5. *Prove that a minimum element is also a minimal element.*

Solution. Let $P = (X, \preccurlyeq)$ be a poset and let $a \in X$ be a minimum element. Suppose a is not a minimal element. Then, there exists $b \neq a$ such that $b \preccurlyeq a$. But since a is minimum we have $a \preccurlyeq b$. Therefore, by anti-symmetry we have $a = b$. This contradicts $b \neq a$ so our supposition must be wrong and a is a minimal element. □

Problem 6. *Prove or disprove: If a poset has a single minimal element, then it is also the minimum element.*

Solution. Consider the usual order $\leqslant$ on $\mathbb{Z} \setminus \{0\}$. Define an ordering $\preccurlyeq$ on $\mathbb{Z}$ to be this order together with 0 being related to all positive integers. That is $a \preccurlyeq b$ if

- either $a \neq 0, b \neq 0, a \leqslant b$,
- or $a = 0, b \geqslant 0$.

This ordering has a single minimal element: zero, and no minimum element. □

Problem 7. *Prove by mathematical induction that a finite poset always has a maximal element.*

Solution. The statement only holds for finite non-empty posets.

Base case: Let $(X, \preceq)$ be a poset with $|X| = 1$. Then the only element in X is also a maximal element.

Inductive step: Suppose any poset with n elements has at least one maximal element.

Let $P = (X, \preccurlyeq)$ be a poset with $|X| = n + 1$ for $n \geqslant 1$. Let $a \in X$. Consider the poset $P' = (X \setminus \{a\}, \preccurlyeq')$ where $\preccurlyeq'$ is the restriction of $\preccurlyeq$ to $X \setminus \{a\}$. By inductive hypothesis P' has a maximal element, say b.

If a and b are incomparable or if $a \preccurlyeq b$ then b is maximal in P. Otherwise, suppose b is not maximal in P. Then $\exists c \in X : c \neq b \wedge b \preccurlyeq c$. But $c \notin X \setminus \{a\}$ otherwise b is not maximal in P'. So $c = a$ and $b \preccurlyeq a$ which contradicts the fact that a, b are incomparable (by definition of incomparability) or $a \preccurlyeq b$ (by anti-symmetry).

If $b \preccurlyeq a$ then a is maximal in P. Otherwise $\exists c \in X \setminus \{a\} : a \preccurlyeq c$. Also, $b \neq c$ otherwise we have $b \preccurlyeq a \wedge a \preccurlyeq b, a \neq b$ contradicting anti-symmetry. But then by transitivity $b \preccurlyeq c$ and so b is not maximal in P', a contradiction.

□

Problem 8. *Let R be a relation over a set X, and let $Y \subseteq X$.*

1. *Prove that if R is an ordering over X then $R \cap (Y \times Y)$ is an ordering over Y.*
2. *Prove that if R is an equivalence relation over X then $R \cap (Y \times Y)$ is an equivalence relation over Y.*

Solution. We only prove the first part. The proof of the second part is analogous.

In order to prove that $S = R \cap (Y \times Y)$ is an ordering we need to prove that S is reflexive,antisymmetric and transitive.

Let $y \in Y$ we need to show that $(y, y) \in S$. But $(y, y) \in R$ since R is an ordering and $(y, y) \in Y \times Y$ obviously, hence S is reflexive.

To show that S is antisymmetric, assume that $(a, b), (b, a) \in S$, then $(a, b), (b, a) \in R$ and since R is antisymmetric $a = b$.

Lastly we need to prove that S is transitive. Let $a, b, c \in Y$ be such that $(a, b), (b, c) \in S \subseteq R$. Then it follows that $(a, c) \in R$. Also, $(a, c) \in Y \times Y$, therefore $(a, c) \in S$. □

Problem 9. *Let R and S be arbitrary equivalence relations (resp. orderings) on the set X. Determine which of the following are necessarily equivalence (resp. ordering):*

1. *$R \cap S$*

2. $R \cup S$

3. $R \setminus S$

4. $R \circ S$

Solution.

1. Suppose R, S are reflexive. Let $x \in X$. Since R, S are reflexive $(x, x) \in R, (x, x) \in S$. So $(x, x) \in R \cap S$ and $R \cap S$ is reflexive.

   Suppose R, S are symmetric. Let $(x, y) \in R \cap S$. Then $(x, y) \in R, (x, y) \in S$. Since R, S are symmetric $(y, x) \in R, (y, x) \in S$. So $(y, x) \in R \cap S$ and $R \cap S$ is symmetric.

   Suppose R, S are anti-symmetric. Let $(x, y), (y, x) \in R \cap S$. So $(x, y), (y, x) \in R$. Since R is antisymmetric, we have $x = y$. Therefore, $R \cap S$ is antisymmetric.

   Suppose R, S are transitive. Let $(x, y), (y, z) \in R \cap S$. So $(x, y), (y, z) \in R$ and $(x, y), (y, z) \in S$. Since R, S are transitive $(x, z) \in R, (x, z) \in S$. So $(x, z) \in R \cap S$. Therefore, $R \cap S$ is transitive.

   This shows that $R \cap S$ inherits the property of reflexivity, symmetry, anti-symmetry, and transitivity if both R and S have that property. Therefore, if R, S is an ordering, so is $R \cap S$, and if R, S are equivalences, so is $R \cap S$.

2. Let $X = \{1, 2, 3\}$. Let $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$ and $S = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$ then both R and S are equivalence relations but $R \cup S = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$ is not transitive and hence not an equivalence.

   Let $X = \{1, 2, 3\}$. Let $R = \{(1, 1), (2, 2), (3, 3), (1, 2)\}$ and $S = \{(1, 1), (2, 2), (3, 3), (2, 3)\}$ then both R and S are orderings but $R \cup S = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$ is not transitive and hence not an ordering.

   This shows that $R \cup S$ does not necessarily inherit the properties being equivalence or ordering if both R, S have that property.

3. Let X be any non-empty set. Then for any two equivalence relations (resp. orderings) over X, $R \setminus S$ is not reflexive so it is not an equivalence (resp. ordering). If $X = \emptyset$ then the only relation that can be defined on it is the empty relation which is both equivalence and an ordering over $\emptyset$.

4. Let $X = \{1, 2, 3\}$. Let $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$ and $S = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$ then both R and S are equivalence relations but $R \circ S = \{(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 1), (2, 3), (3, 2)$ is not symmetric and hence not an equivalence relation.

   Let $X = \{1, 2, 3\}$. Let $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$ and $S = \{(1, 1), (2, 2), (3, 3), (3, 1)\}$ then both R and S are orderings but $R \circ S = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 1), (1, 3)\}$ is not anti-symmetric and hence not and ordering.

   This shows that $R \circ S$ does not necessarily inherit the property of being an equivalence (resp. ordering) even if R and S both have the property.

□

Problem 10. *Prove that every partial ordering on a finite set X has at least one linear extension. For a partial ordering $\preccurlyeq$, linear extension is any linear order $\leq$ such that $x \preccurlyeq y \implies x \leq y$.*

Solution. (The following proof is Theorem 2.2.1 of the book Invitation to Discrete Mathematics)

We proceed by induction on $|X|$. For $|X| = 0$ the only relation, the empty relation is already a linear order. Thus, let us consider an ordered set $(X, \preccurlyeq)$ with $|X| \geq 1$. Let $x_0 \in X$ be a minimal element in $(X, \preccurlyeq)$. We set $X' = X \setminus \{x_0\}$, and we let $\preccurlyeq'$ be the relation $\preccurlyeq$ restricted to the set X'. We already know that $(X', \preccurlyeq')$ is an ordered set, and hence by the inductive hypothesis, there exists a linear ordering $\leq'$ of X' such that $x \preccurlyeq' y$ implies $x \leq' y$ for all $x, y \in X'$. We define a relation $\leq$ on the set X as follows:

$$x_0 \leq y \text{ for each } y \in X;$$
$$x \leq y \text{ whenever } x \leq' y.$$

Obviously, $x \preccurlyeq y$ implies $x \leq y$. The reader is invited to verify that $\leq$ is indeed a linear ordering. □