
Problem 1. *Let $1 \leq r \leq n$. How many sequences $(a_1, \ldots, a_r)$, with each $a_i \in \{1, \ldots, n\}$, satisfy:*

1. *$a_1 < a_2 < \cdots < a_r$?*
2. *$a_1 \leq a_2 \leq \cdots \leq a_r$?*

Problem 2. *There are n seats in a row. In how many ways can we choose k seats so that no two chosen seats are adjacent? Assume $k \geq 1$ and $n \geq 2k - 1$.*

Problem 3. *Prove that, for every integer $n \geq 2$,*

$$\left(\frac{n}{2}\right)^{n/2} \leq n! \leq \left(\frac{n+1}{2}\right)^{n}.$$

Use your estimates to show that, for every fixed real number $c > 0$, we have $n! > c^n$ for all sufficiently large n.

Problem 4.

1. *Prove that the largest of the numbers $\binom{2n}{0}, \binom{2n}{1}, \ldots, \binom{2n}{2n}$ is $\binom{2n}{n}$.*
2. *Deduce that $\frac{4^n}{2n+1} \leq \binom{2n}{n} \leq 4^n$.*

Problem 5. *Guillermo moved to a new country and has to get a new phone number. The carrier offers him a possibility to choose his own number but they have a weird rule that each odd digit needs to be contained in the number. How many 10-digit phone numbers can Guillermo make that satisfy this rule?*

Problem 6. *Use inclusion-exclusion principle to calculate the number of surjective functions from X to Y where $|X| = n$ and $|Y| = m$.*

Problem 7. *Use inclusion-exclusion principle to determine the number of integer solutions of the equation*

$$x_1 + x_2 + x_3 + x_4 = 20$$

satisfying the constraints $1 \leq x_1 \leq 6; 0 \leq x_2 \leq 7; 4 \leq x_3 \leq 8; 2 \leq x_4 \leq 6$.

Problem 8. *Let D_m denote the number of permutations of $\{1, \ldots, m\}$ with no fixed points, with $D_0 = 1$.*

1. *How many permutations of $\{1, \ldots, n\}$ have exactly k fixed points?*
2. *Give a combinatorial proof of*

$$\sum_{k=0}^{n} \binom{n}{k} D_{n-k} = n!.$$

3. *Explain directly why a permutation cannot have exactly $n - 1$ fixed points.*