
Problem 1. *In a group of 120 students, each student is assigned a unique ID number from 1 to 120. We define an equivalence relation on this set of students as follows: two students are related if and only if the difference between their ID numbers is divisible by 4. How many equivalence classes does this relation create, and what is the size of each equivalence class?*

Problem 2. *Consider a poset $(P, \preceq)$ where $P = \{1, 2, 3, 4, 5, 6\}$ and the partial order is defined by the divisibility relation ($a \preceq b$ if and only if a divides b). Draw the Hasse diagram for this poset.*

1. *Identify all maximal and minimal elements of the poset.*
2. *Find the length of the longest chain and the width of the poset.*
3. *Verify that for this poset, $\alpha(P)\omega(P) \geq |P|$, where $\alpha(P)$ is the size of the largest antichain and $\omega(P)$ is the size of the largest chain.*

Problem 3. *Assume that we are given 26 distinct positive integers. Show that either there exist 6 of them $x_1 < x_2 < x_3 < x_4 < x_5 < x_6$, with x_i dividing x_{i+1} for $i \in [5]$ or there exist six of them, with no one dividing another one of these six.*

Problem 4. *Let R be a relation over a set X. Consider the relation $\preceq_R$ defined over $X \times X$ as follows: $(a_1, b_1) \preceq_R (a_2, b_2)$ if either $a_1 \neq a_2 \wedge (a_1, a_2) \in R$ or $a_1 = a_2 \wedge (b_1, b_2) \in R$. Prove that $\preceq_R$ is an order over $X \times X$ if and only if R is an order over X.*

Problem 5. *Let X be a set with n elements. Determine the number of relations R over X such that*

1. *R is arbitrary.*
2. *R is reflexive.*
3. *R is symmetric.*
4. *R is anti-symmetric.*

Problem 6. *Decide which of the following functions $f : \mathbb{Z} \to \mathbb{Z}$ are injective, which are surjective and which are bijective.*

1. $x \to 1 + x^2$
2. $x \to 1 + x^3$
3. $x \to 1 + x^2 + x^3$

Problem 7. *Let X be a finite set and $f : X \to X$ be a function. Prove that f is an injection if and only if f is a bijection.*

Problem 8. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be an n-element set and let $Y = \{0,1\}^n$, where $A^n = \underbrace{A \times A \times \cdots \times A}_{n \text{ times}}$. Give a bijection $f : 2^X \to Y$ and deduce that the number of subsets of an n-element set is 2^n.*

Problem 9. *Let X be an n-element set with $n \geqslant 1$. Let $\mathcal{E}$ be the set of even-cardinality subsets of X and $\mathcal{O}$ be the set of odd cardinality subsets of X. Let $a \in X$ and define $f : \mathcal{E} \to \mathcal{O}$ as follows: $f(S) = S \triangle \{a\}$. Prove that f is a bijection. Deduce that the number of even-cardinality subsets of a set with n elements is 2^{n-1}.*

Problem 10. *Prove that the number of ordred tuples $(A_1, \ldots, A_r)$ such that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_r \subseteq [n]$ is $(r+1)^n$ by giving a bijection from the set of such ordered pairs to $\{0, \ldots, r\}^n$. (Also prove that the function you provide is indeed a bijection.)*