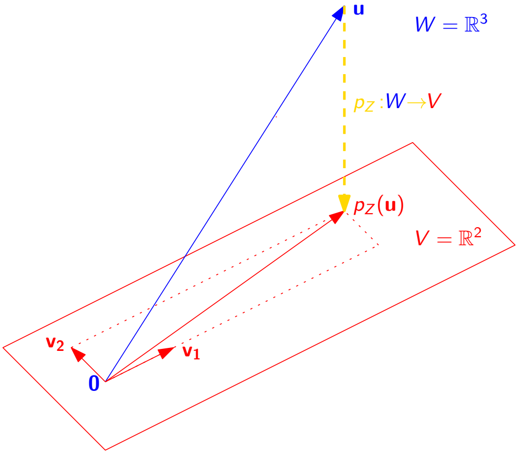
Orthogonal projection

Definition: Let W be an inner space and V its *subspace* with orthonormal basis $Z = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$. The map $p_Z : W \to V$ defined as $p_Z(\boldsymbol{u}) = \sum_{i=1}^{n} \langle \boldsymbol{u} | \boldsymbol{v}_i \rangle \boldsymbol{v}_i$ is the *orthogonal projection* of W onto V.

Example:



Observation: Any orthogonal projection is a linear map.

Proof:

$$p_Z(a\boldsymbol{u}) = \sum_{i=1}^{n} \langle a\boldsymbol{u}|\boldsymbol{v}_i\rangle \boldsymbol{v}_i = \sum_{i=1}^{n} a\langle \boldsymbol{u}|\boldsymbol{v}_i\rangle \boldsymbol{v}_i = a\sum_{i=1}^{n} \langle \boldsymbol{u}|\boldsymbol{v}_i\rangle \boldsymbol{v}_i = ap_Z(\boldsymbol{u})$$

$$p_Z(\boldsymbol{u}+\boldsymbol{w}) = \sum_{i=1}^{n} \langle \boldsymbol{u}+\boldsymbol{w}|\boldsymbol{v}_i\rangle \boldsymbol{v}_i = \sum_{i=1}^{n} (\langle \boldsymbol{u}|\boldsymbol{v}_i\rangle + \langle \boldsymbol{w}|\boldsymbol{v}_i\rangle)\boldsymbol{v}_i =$$

$$\sum_{i=1}^{n} \langle \boldsymbol{u}|\boldsymbol{v}_i\rangle \boldsymbol{v}_i + \sum_{i=1}^{n} \langle \boldsymbol{w}|\boldsymbol{v}_i\rangle \boldsymbol{v}_i = p_Z(\boldsymbol{u}) + p_Z(\boldsymbol{w})$$

Observation: Any orthogonal projection is a linear map.

Lemma: Let p_Z be an orthogonal projection of W onto V, then $\boldsymbol{u} - p_Z(\boldsymbol{u}) \perp \boldsymbol{v}_i$ for any $\boldsymbol{v}_i \in Z$.

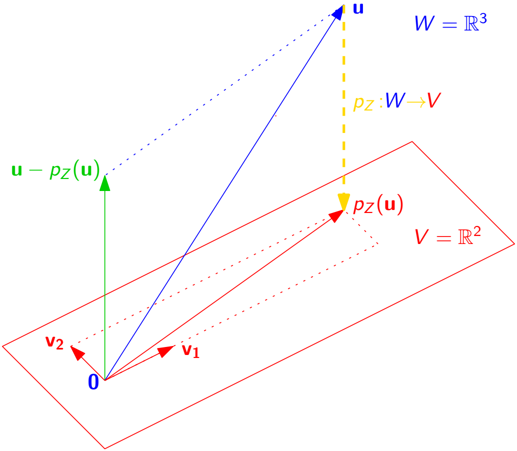
Proof:

$\langle \boldsymbol{u} - p_Z(\boldsymbol{u}) | \boldsymbol{v}_i \rangle =$

$\left\langle \boldsymbol{u} - \sum_{j=1}^{n} \langle \boldsymbol{u} | \boldsymbol{v}_j \rangle \boldsymbol{v}_j \middle| \boldsymbol{v}_i \right\rangle =$

$\langle \boldsymbol{u} | \boldsymbol{v}_i \rangle - \sum_{j=1}^{n} \langle \boldsymbol{u} | \boldsymbol{v}_j \rangle \langle \boldsymbol{v}_j | \boldsymbol{v}_i \rangle =$

$\langle \boldsymbol{u} | \boldsymbol{v}_i \rangle - \langle \boldsymbol{u} | \boldsymbol{v}_i \rangle = 0$

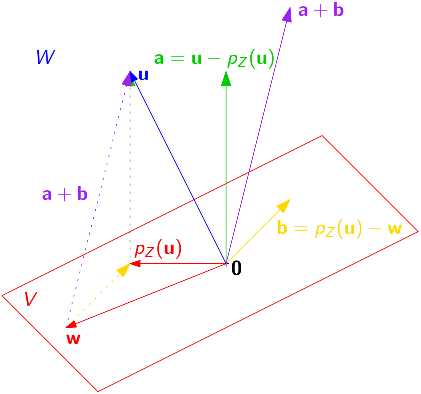


Projection and distance

Observation: The vector $p_Z(\boldsymbol{u})$ is the vector from $V = \mathcal{L}(Z)$ which is nearest to u, in the sense that it minimizes $||\boldsymbol{u} - p_Z(\boldsymbol{u})||$.

Proof: For any $\boldsymbol{w} \in V$, $\boldsymbol{w} \neq p_Z(\boldsymbol{u})$
let $\boldsymbol{a} = \boldsymbol{u} - p_Z(\boldsymbol{u})$,
$\boldsymbol{b} = p_Z(\boldsymbol{u}) - \boldsymbol{w} \neq \boldsymbol{0}$.
Since $\boldsymbol{b} \in V$, we get $\langle \boldsymbol{a} | \boldsymbol{b} \rangle = 0$.

Now: $||\boldsymbol{u} - \boldsymbol{w}|| = ||\boldsymbol{a} + \boldsymbol{b}||$
$= \sqrt{\langle \boldsymbol{a} + \boldsymbol{b} | \boldsymbol{a} + \boldsymbol{b} \rangle}$
$= \sqrt{\langle \boldsymbol{a} | \boldsymbol{a} \rangle + \langle \boldsymbol{a} | \boldsymbol{b} \rangle + \langle \boldsymbol{b} | \boldsymbol{a} \rangle + \langle \boldsymbol{b} | \boldsymbol{b} \rangle}$
$= \sqrt{\langle \boldsymbol{a} | \boldsymbol{a} \rangle + \langle \boldsymbol{b} | \boldsymbol{b} \rangle}$
$> \sqrt{\langle \boldsymbol{a} | \boldsymbol{a} \rangle}$
$= ||\boldsymbol{a}|| = ||\boldsymbol{u} - p_Z(\boldsymbol{u})||$



Corollary: The map p_Z is independent on the choice of the basis Z.

Approximate solution of non-solvable systems

Observation: The vector $p_Z(\boldsymbol{u})$ is the vector from $V = \mathcal{L}(Z)$ which is nearest to u, in the sense that it minimizes $||\boldsymbol{u} - p_Z(\boldsymbol{u})||$.

If a system $\boldsymbol{Ax} = \boldsymbol{b}$ has no solution, i.e. when $\boldsymbol{b} \notin \mathcal{C}(\boldsymbol{A})$, then we may project $\boldsymbol{b}$ into $\mathcal{C}(\boldsymbol{A})$ and get $\boldsymbol{b}'$.

The system $\boldsymbol{Ax} = \boldsymbol{b}'$ now has a solution. By the observation such $\boldsymbol{x}$ minimizes the error $||\boldsymbol{b} - \boldsymbol{b}'|| = ||\boldsymbol{b} - \boldsymbol{Ax}||$.

This is the principle of the so called *method of least squares*.

Calculation options:

- Get an orthonormal basis $\mathcal{C}(\boldsymbol{A})$ and project $\boldsymbol{b}$ to $\boldsymbol{b}'$, or
- Instead of $\boldsymbol{Ax} = \boldsymbol{b}'$ solve equivalent $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$.

Proof: $\boldsymbol{b}$ projects to $\boldsymbol{b}' \in \mathcal{C}(\boldsymbol{A}) \iff \boldsymbol{b} - \boldsymbol{b}' \in \mathcal{C}(\boldsymbol{A})^\perp = \ker(\boldsymbol{A}^T)$
$\iff \quad \boldsymbol{A}^T(\boldsymbol{b} - \boldsymbol{b}') = \boldsymbol{0} \quad \iff \quad \boldsymbol{A}^T\boldsymbol{b} = \boldsymbol{A}^T\boldsymbol{b}' = \boldsymbol{A}^T\boldsymbol{Ax}$

Gram-Schmidt orthonormalization

A process that transfers any basis $(\boldsymbol{u}_1, \dots \boldsymbol{u}_n)$ of an inner space V to an orthonormal basis $(\boldsymbol{v}_1, \dots \boldsymbol{v}_n)$:

for $i = 1, \dots, n$ **do**

$$\boldsymbol{w}_i = \boldsymbol{u}_i - \sum_{j=1}^{i-1} \langle \boldsymbol{u}_i | \boldsymbol{v}_j \rangle \boldsymbol{v}_j$$

$$\boldsymbol{v}_i = \frac{1}{||\boldsymbol{w}_i||} \boldsymbol{w}_i$$

end

Gram-Schmidt orthonormalization — example

for $i = 1, \ldots, n$ **do**

$$\boldsymbol{w}_i = \boldsymbol{u}_i - \sum_{j=1}^{i-1} \langle \boldsymbol{u}_i | \boldsymbol{v}_j \rangle \boldsymbol{v}_j$$

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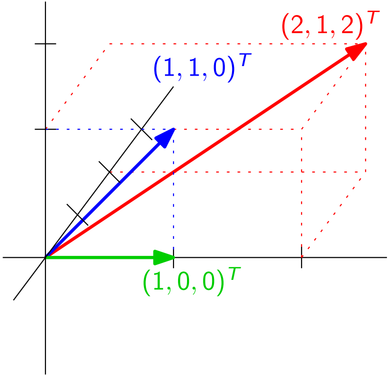
end



Gram-Schmidt orthonormalization — example

for $i = 1, \ldots, n$ **do**

$$\boldsymbol{w}_i = \boldsymbol{u}_i - \sum_{j=1}^{i-1} \langle \boldsymbol{u}_i | \boldsymbol{v}_j \rangle \boldsymbol{v}_j$$

$$\boldsymbol{v}_i = \frac{1}{||\boldsymbol{w}_i||} \boldsymbol{w}_i$$

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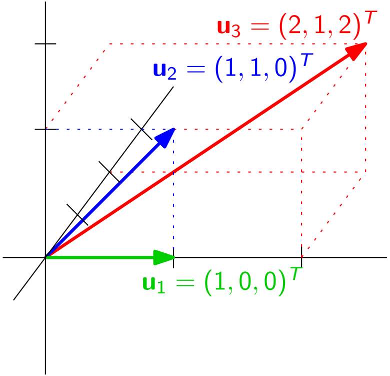


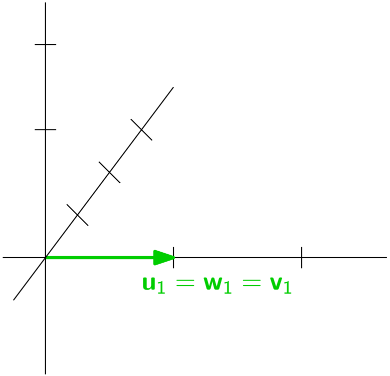
Gram-Schmidt orthonormalization — example

for $i = 1, \ldots, n$ **do**

$$\boldsymbol{w}_i = \boldsymbol{u}_i - \sum_{j=1}^{i-1} \langle \boldsymbol{u}_i | \boldsymbol{v}_j \rangle \boldsymbol{v}_j$$

$$\boldsymbol{v}_i = \frac{1}{||\boldsymbol{w}_i||} \boldsymbol{w}_i$$

end



$i = 1:$

$$\boldsymbol{w}_1 = \boldsymbol{u}_1 - \sum_{j=1}^{0} \langle \boldsymbol{u}_i | \boldsymbol{v}_j \rangle \boldsymbol{v}_j = \boldsymbol{u}_1 = (1, 0, 0)^T$$

$$\boldsymbol{v}_1 = \frac{1}{||\boldsymbol{w}_1||} \boldsymbol{w}_1 = \frac{1}{1}(1, 0, 0)^T = (1, 0, 0)^T$$

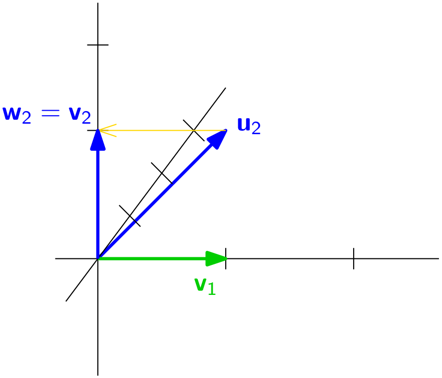
Gram-Schmidt orthonormalization — example

for $i = 1, \ldots, n$ **do**

$\boldsymbol{w}_i = \boldsymbol{u}_i - \sum_{j=1}^{i-1} \langle \boldsymbol{u}_i | \boldsymbol{v}_j \rangle \boldsymbol{v}_j$

$\boldsymbol{v}_i = \frac{1}{||\boldsymbol{w}_i||} \boldsymbol{w}_i$

end



$i = 2:$

$\boldsymbol{w}_2 = \boldsymbol{u}_2 - \langle \boldsymbol{u}_2 | \boldsymbol{v}_1 \rangle \boldsymbol{v}_1 = (1, 1, 0)^T - 1 \cdot (1, 0, 0)^T = (0, 1, 0)^T$

$\boldsymbol{v}_2 = \frac{1}{||\boldsymbol{w}_2||} \boldsymbol{w}_2 = \frac{1}{1}(0, 1, 0)^T = (0, 1, 0)^T$

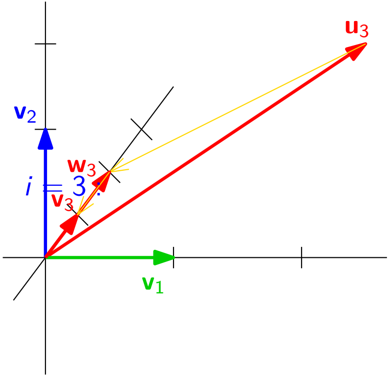
Gram-Schmidt orthonormalization — example

for $i = 1, \ldots, n$ **do**

$$\boldsymbol{w}_i = \boldsymbol{u}_i - \sum_{j=1}^{i-1} \langle \boldsymbol{u}_i | \boldsymbol{v}_j \rangle \boldsymbol{v}_j$$

$$\boldsymbol{v}_i = \frac{1}{||\boldsymbol{w}_i||} \boldsymbol{w}_i$$

end



$i = 3$:

$$\boldsymbol{w}_3 = \boldsymbol{u}_3 - \langle \boldsymbol{u}_3 | \boldsymbol{v}_1 \rangle \boldsymbol{v}_1 - \langle \boldsymbol{u}_3 | \boldsymbol{v}_2 \rangle \boldsymbol{v}_2 =$$
$$= (2, 1, 2)^T - 2 \cdot (1, 0, 0)^T - 1 \cdot (0, 1, 0)^T = (0, 0, 2)^T$$
$$\boldsymbol{v}_3 = \frac{1}{||\boldsymbol{w}_3||} \boldsymbol{w}_3 = \frac{1}{2}(0, 0, 2)^T = (0, 0, 1)^T$$

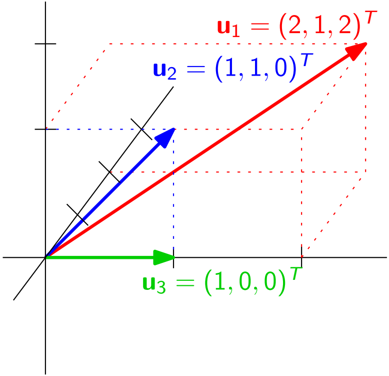
Gram-Schmidt orthonormalization — another order

for $i = 1, \ldots, n$ **do**

$$\boldsymbol{w}_i = \boldsymbol{u}_i - \sum_{j=1}^{i-1} \langle \boldsymbol{u}_i | \boldsymbol{v}_j \rangle \boldsymbol{v}_j$$

$$\boldsymbol{v}_i = \frac{1}{||\boldsymbol{w}_i||} \boldsymbol{w}_i$$

end



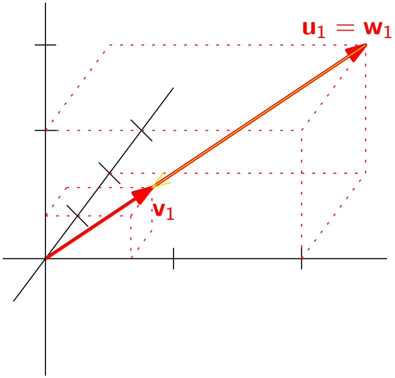
Gram-Schmidt orthonormalization — another order

for $i = 1, \ldots, n$ **do**

$$\boldsymbol{w}_i = \boldsymbol{u}_i - \sum_{j=1}^{i-1} \langle \boldsymbol{u}_i | \boldsymbol{v}_j \rangle \boldsymbol{v}_j$$

$$\boldsymbol{v}_i = \frac{1}{||\boldsymbol{w}_i||} \boldsymbol{w}_i$$

end



$i = 1$:

$\boldsymbol{w}_1 = \boldsymbol{u}_1 = (2, 1, 2)^T$

$\boldsymbol{v}_1 = \frac{1}{||\boldsymbol{w}_1||} \boldsymbol{w}_1 = \frac{1}{3}(2, 1, 2)^T = (\frac{2}{3}, \frac{1}{3}, \frac{2}{3})^T$

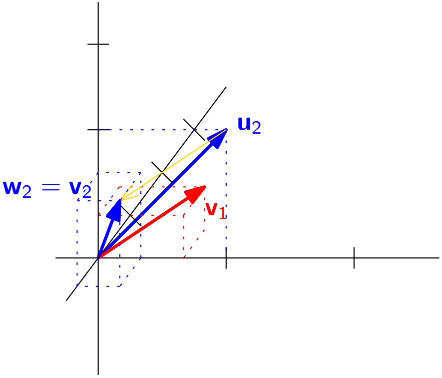
Gram-Schmidt orthonormalization — another order

for $i = 1, \ldots, n$ **do**

$$\boldsymbol{w}_i = \boldsymbol{u}_i - \sum_{j=1}^{i-1} \langle \boldsymbol{u}_i | \boldsymbol{v}_j \rangle \boldsymbol{v}_j$$

$$\boldsymbol{v}_i = \frac{1}{||\boldsymbol{w}_i||} \boldsymbol{w}_i$$

end



$i = 2:$

$$\boldsymbol{w}_2 = \boldsymbol{u}_2 - \langle \boldsymbol{u}_2 | \boldsymbol{v}_1 \rangle \boldsymbol{v}_1 = (1, 1, 0)^T - 1 \cdot (\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3})^T = (\tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{2}{3})^T$$

$$\boldsymbol{v}_2 = \frac{1}{||\boldsymbol{w}_2||} \boldsymbol{w}_2 = \frac{1}{1} (\tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{2}{3})^T$$

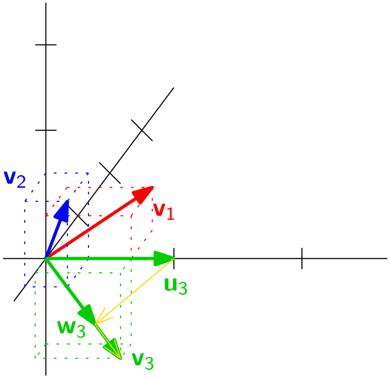
Gram-Schmidt orthonormalization — another order

for $i = 1, \ldots, n$ **do**

$$\boldsymbol{w}_i = \boldsymbol{u}_i - \sum_{j=1}^{i-1} \langle \boldsymbol{u}_i | \boldsymbol{v}_j \rangle \boldsymbol{v}_j$$

$$\boldsymbol{v}_i = \frac{1}{||\boldsymbol{w}_i||} \boldsymbol{w}_i$$

end



$i = 3:$

$$\boldsymbol{w}_3 = \boldsymbol{u}_3 - \langle \boldsymbol{u}_3 | \boldsymbol{v}_1 \rangle \boldsymbol{v}_1 - \langle \boldsymbol{u}_3 | \boldsymbol{v}_2 \rangle \boldsymbol{v}_2 =$$
$$= (1, 0, 0)^T - \tfrac{2}{3} \cdot (\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3})^T - \tfrac{1}{3} \cdot (\tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{2}{3})^T = (\tfrac{4}{9}, -\tfrac{4}{9}, -\tfrac{2}{9})^T$$

$$\boldsymbol{v}_3 = \frac{1}{||\boldsymbol{w}_3||} \boldsymbol{w}_3 = \frac{1}{2/3} (\tfrac{4}{9}, -\tfrac{4}{9}, -\tfrac{2}{9})^T = (\tfrac{2}{3}, -\tfrac{2}{3}, -\tfrac{1}{3})^T$$

Gram-Schmidt orthonormalization

A process that transfers any basis $(\boldsymbol{u}_1, \dots \boldsymbol{u}_n)$ of an inner space V to an orthonormal basis $(\boldsymbol{v}_1, \dots \boldsymbol{v}_n)$:

for $i = 1, \dots, n$ **do**

1. $\boldsymbol{w}_i = \boldsymbol{u}_i - \sum_{j=1}^{i-1} \langle \boldsymbol{u}_i | \boldsymbol{v}_j \rangle \boldsymbol{v}_j$
2. $\boldsymbol{v}_i = \frac{1}{||\boldsymbol{w}_i||} \boldsymbol{w}_i$

end

Correctness:

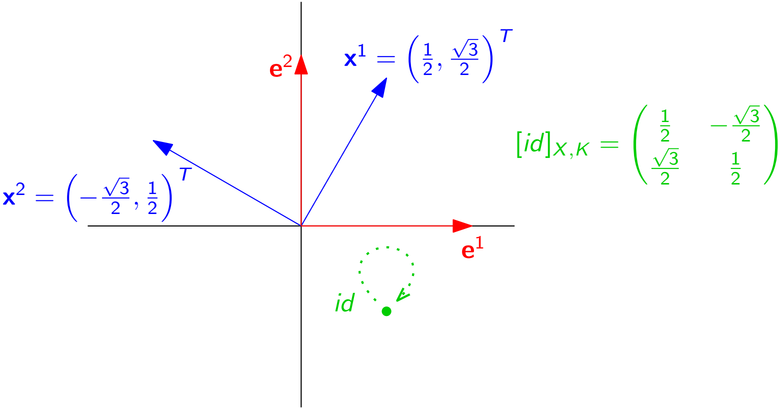
- Due to 1. and the previous lemma: $\boldsymbol{w}_i \perp \boldsymbol{v}_j$ for each $j < i$, hence $\boldsymbol{v}_i \perp \boldsymbol{v}_j$ whenever $j \neq i$
- Due to 2.: $||\boldsymbol{v}_i|| = \left|\left| \frac{1}{||\boldsymbol{w}_i||} \boldsymbol{w}_i \right|\right| = \frac{||\boldsymbol{w}_i||}{||\boldsymbol{w}_i||} = 1$.
- Due to the exchange lemma: $\mathcal{L}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_{i-1}, \boldsymbol{u}_i) = \mathcal{L}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_{i-1}, \boldsymbol{w}_i) = \mathcal{L}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_i)$

Consequence: Let V be a subspace of an inner space W. Then any orthonormal basis of V has an orthonormal extension to W.

Linear maps that preserve the inner product

Definition: A linear map f between inner spaces V and W is *isometry* if it preserves the inner product, i.e. $\langle \boldsymbol{u} | \boldsymbol{w} \rangle = \langle f(\boldsymbol{u}) | f(\boldsymbol{w}) \rangle$.

Example: The identity preserves the inner product.



Linear maps that preserve the inner product

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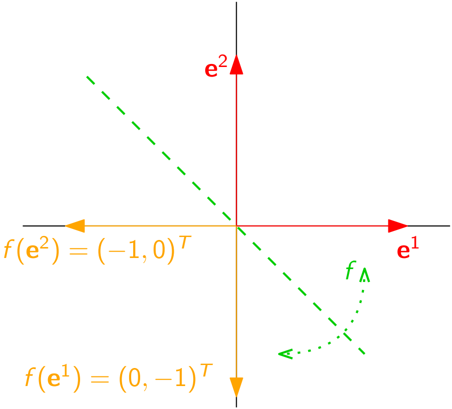
Example: Axis symmetry preserves the inner product.



$$[f]_{K,K} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Linear maps that preserve the inner product

Definition: A linear map f between inner spaces V and W is *isometry* if it preserves the inner product, i.e. $\langle \boldsymbol{u}|\boldsymbol{w}\rangle = \langle f(\boldsymbol{u})|f(\boldsymbol{w})\rangle$.

Theorem: A linear map between inner spaces V and W is isometry *if and only if* it preserves the associated norm, i.e. $||\boldsymbol{u}|| = ||f(\boldsymbol{u})||$.

Proof: As the norm depends on the inner product, $\Rightarrow$ follows.

$\Leftarrow$ compare:

$$||\boldsymbol{u}+a\boldsymbol{w}||^2 = ||\boldsymbol{u}||^2 + a\langle \boldsymbol{w}|\boldsymbol{u}\rangle + \bar{a}\langle \boldsymbol{u}|\boldsymbol{w}\rangle + a\bar{a}||\boldsymbol{w}||^2$$

$$||$$

$$||f(\boldsymbol{u}+a\boldsymbol{w})||^2 = ||f(\boldsymbol{u})||^2 + a\langle f(\boldsymbol{w})|f(\boldsymbol{u})\rangle + \bar{a}\langle f(\boldsymbol{u})|f(\boldsymbol{w})\rangle + a\bar{a}||f(\boldsymbol{w})||^2$$

for $a=1$ we get: $\langle \boldsymbol{w}|\boldsymbol{u}\rangle + \langle \boldsymbol{u}|\boldsymbol{w}\rangle = \langle f(\boldsymbol{w})|f(\boldsymbol{u})\rangle + \langle f(\boldsymbol{u})|f(\boldsymbol{w})\rangle$

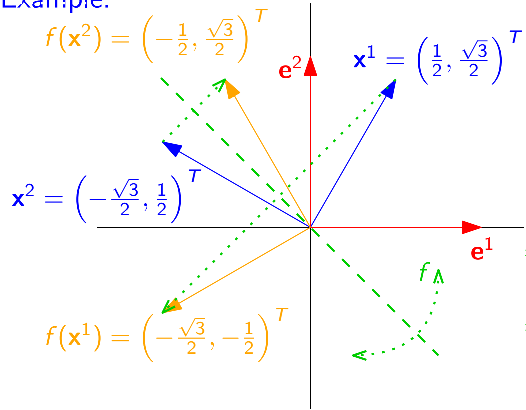
for $a=i$ we get: $\langle \boldsymbol{w}|\boldsymbol{u}\rangle - \langle \boldsymbol{u}|\boldsymbol{w}\rangle = \langle f(\boldsymbol{w})|f(\boldsymbol{u})\rangle - \langle f(\boldsymbol{u})|f(\boldsymbol{w})\rangle$

$$\Rightarrow \langle \boldsymbol{u}|\boldsymbol{w}\rangle = \langle f(\boldsymbol{u})|f(\boldsymbol{w})\rangle$$

Matrix characterization of bijective isometries

Theorem: Let V and W be inner spaces of finite dimension and let X and Y be their orthonormal bases.
A linear map $f : V \to W$ is a bijective isometry iff $[f]_{XY}$ is unitary.

Example:



$$[f]_{X,K} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$= [f]_{K,K}[id]_{X,K}$$

Observe that the product of unitary matrices is unitary.

Matrix characterization of bijective isometries

Theorem: Let V and W be inner spaces of finite dimension and let X and Y be their orthonormal bases.
A linear map $f : V \to W$ is a bijective isometry iff $[f]_{XY}$ is unitary.

Proof: Linear bijective implies equal dimensions and vice versa.
Since X is orthonormal: $\langle \boldsymbol{u} | \boldsymbol{w} \rangle = [\boldsymbol{w}]_X^H [\boldsymbol{u}]_X$
Since Y is orthonormal: $\langle f(\boldsymbol{u}) | f(\boldsymbol{w}) \rangle = [f(\boldsymbol{w})]_Y^H [f(\boldsymbol{u})]_Y$

$$= [\boldsymbol{w}]_X^H [f]_{XY}^H [f]_{XY} [\boldsymbol{u}]_X$$

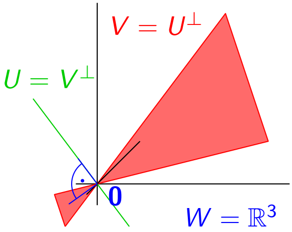
Note that the matrix identity $\boldsymbol{x}^T \boldsymbol{y} = \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{y}$ holds for all suitable vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ only if $\boldsymbol{A}$ is the identity matrix.

In our case, f is isometry
iff $[\boldsymbol{w}]_X^H [\boldsymbol{u}]_X = [\boldsymbol{w}]_X^H [f]_{XY}^H [f]_{XY} [\boldsymbol{u}]_X$ holds for all $\boldsymbol{u}$ and $\boldsymbol{w}$,
which holds if and only if $[f]_{XY}^H [f]_{XY} = \boldsymbol{I}$,
i.e. when $[f]_{XY}$ is unitary.

Orthogonal complement

Definition: Let V be a subset of an inner space W. The *orthogonal complement* of V is the set $V^\perp = \{\boldsymbol{u} \in W : \forall \boldsymbol{v} \in V : \boldsymbol{u} \perp \boldsymbol{v}\}$.

Example:



Observation: If $U \subseteq V$ then $U^\perp \supseteq V^\perp$.

Proof: $$V^\perp = \{\boldsymbol{u} \in W : \forall \boldsymbol{v} \in V : \boldsymbol{u} \perp \boldsymbol{v}\} \subseteq \{\boldsymbol{u} \in W : \forall \boldsymbol{v} \in U : \boldsymbol{u} \perp \boldsymbol{v}\} = U^\perp$$

Orthogonal complement

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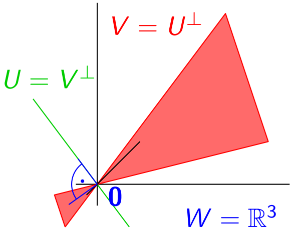
Example:



Observation: If $U \subseteq V$ then $U^\perp \supseteq V^\perp$.

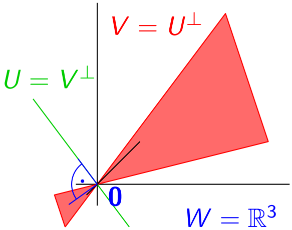
Observation: Each orthogonal complement is a subspace of W.

Proof: $\boldsymbol{u} \perp \boldsymbol{v} \Longrightarrow \langle a\boldsymbol{u}|\boldsymbol{v}\rangle = a\langle \boldsymbol{u}|\boldsymbol{v}\rangle = 0 \Longrightarrow (a\boldsymbol{u}) \perp \boldsymbol{v}$
$\boldsymbol{u}, \boldsymbol{w} \perp \boldsymbol{v} \Longrightarrow \langle \boldsymbol{u}+\boldsymbol{w}|\boldsymbol{v}\rangle = \langle \boldsymbol{u}|\boldsymbol{v}\rangle + \langle \boldsymbol{w}|\boldsymbol{v}\rangle = 0 \Longrightarrow (\boldsymbol{u}+\boldsymbol{w}) \perp \boldsymbol{v}$

Orthogonal complement

Definition: Let V be a subset of an inner space W. The *orthogonal complement* of V is the set $V^\perp = \{\boldsymbol{u} \in W : \forall \boldsymbol{v} \in V : \boldsymbol{u} \perp \boldsymbol{v}\}$.

Example:



Observation: If $U \subseteq V$ then $U^\perp \supseteq V^\perp$.

Observation: Each orthogonal complement is a subspace of W.

Observation: For any $V \subseteq W$: $V \cap V^\perp = \{\mathbf{0}\}$

Proof: If $\boldsymbol{u} \in V \cap V^\perp$ then $\langle \boldsymbol{u} | \boldsymbol{u} \rangle = 0$, hence $\boldsymbol{u} = \mathbf{0}$.

For spaces determined by a matrix: $\text{Ker}(\boldsymbol{A}) = (\mathcal{R}(\boldsymbol{A}))^\perp$

For a real matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 3 & 4 & 5 \\ 2 & 6 & 3 & 0 \\ 3 & 9 & 15 & 9 \end{pmatrix} \sim \cdots \sim \begin{pmatrix} 1 & 3 & 4 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Hence $\mathcal{R}(\boldsymbol{A}) = \mathcal{L}\{(1,3,4,5)^T, (0,0,1,-2)^T\} = \mathcal{L}\{\boldsymbol{x}^1, \boldsymbol{x}^2\}$,
and also $\text{Ker}(\boldsymbol{A}) = \mathcal{L}\{(-13,0,2,1)^T, (-3,1,0,0)^T\} = \mathcal{L}\{\boldsymbol{y}^1, \boldsymbol{y}^2\}$.

Prposition: For $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ any $\boldsymbol{u} \in \mathcal{R}(\boldsymbol{A})$ and $\boldsymbol{v} \in \text{Ker}(\boldsymbol{A})$ satisfy $\boldsymbol{u} \perp \boldsymbol{v}$ with respect to the standard inner product.

Example:
$\boldsymbol{u} = \boldsymbol{x}^1 - 2\boldsymbol{x}^2 = (1,3,4,5)^T - 2(0,0,1,-2)^T = (1,3,2,9)^T$
$\boldsymbol{v} = \boldsymbol{y}^1 + 3\boldsymbol{y}^2 = (-13,0,2,1)^T + 3(-3,1,0,0)^T = (-22,3,2,1)^T$

$\langle \boldsymbol{u} | \boldsymbol{v} \rangle = 1 \cdot (-22) + 3 \cdot 3 + 2 \cdot 2 + 9 \cdot 1 = 0$

$$\langle \boldsymbol{u} | \boldsymbol{v} \rangle = \langle \boldsymbol{x}^1 - 2\boldsymbol{x}^2 | \boldsymbol{y}^1 + 3\boldsymbol{y}^2 \rangle = \\ = \langle \boldsymbol{x}^1 | \boldsymbol{y}^1 \rangle + 3\langle \boldsymbol{x}^1 | \boldsymbol{y}^2 \rangle - 2\langle \boldsymbol{x}^2 | \boldsymbol{y}^1 \rangle - 6\langle \boldsymbol{x}^2 | \boldsymbol{y}^2 \rangle = 0$$

For spaces determined by a matrix: $\text{Ker}(\boldsymbol{A}) = (\mathcal{R}(\boldsymbol{A}))^\perp$

For a real matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 3 & 4 & 5 \\ 2 & 6 & 3 & 0 \\ 3 & 9 & 15 & 9 \end{pmatrix} \sim \cdots \sim \begin{pmatrix} 1 & 3 & 4 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Hence $\mathcal{R}(\boldsymbol{A}) = \mathcal{L}\{(1,3,4,5)^T, (0,0,1,-2)^T\} = \mathcal{L}\{\boldsymbol{x}^1, \boldsymbol{x}^2\}$,
and also $\text{Ker}(\boldsymbol{A}) = \mathcal{L}\{(-13,0,2,1)^T, (-3,1,0,0)^T\} = \mathcal{L}\{\boldsymbol{y}^1, \boldsymbol{y}^2\}$.

Prposition: For $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ any $\boldsymbol{u} \in \mathcal{R}(\boldsymbol{A})$ and $\boldsymbol{v} \in \text{Ker}(\boldsymbol{A})$ satisfy $\boldsymbol{u} \perp \boldsymbol{v}$ with respect to the standard inner product.

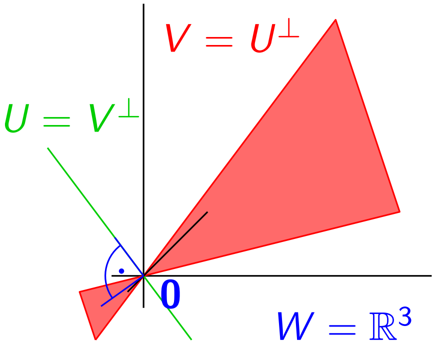
Proof: Denote by $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^r$ a basis of $\mathcal{R}(\boldsymbol{A})$, and similarly by $\boldsymbol{y}^1, \ldots, \boldsymbol{y}^{n-r}$ a basis of $\text{Ker}(\boldsymbol{A})$, where $r = \text{rank}(\boldsymbol{A})$.

Then $\boldsymbol{u} = \sum_{i=1}^{r} a_i \boldsymbol{x}^i$ and $\boldsymbol{v} = \sum_{j=1}^{n-r} b_j \boldsymbol{y}^j$ satisfy

$$\langle \boldsymbol{u} | \boldsymbol{v} \rangle = \left\langle \sum_{i=1}^{r} a_i \boldsymbol{x}_i \middle| \sum_{j=1}^{n-r} b_j \boldsymbol{y}_j \right\rangle = \sum_{i=1}^{r} \sum_{j=1}^{n-r} a_i b_j \langle \boldsymbol{x}_i | \boldsymbol{y}_j \rangle = 0.$$

Properties of the orthogonal complement

Theorem: Each finitely generated inner space W and its subspace V satisfy: $(V^\perp)^\perp = V$ and also $\dim(V) + \dim(V^\perp) = \dim(W)$.



Properties of the orthogonal complement

Theorem: Each finitely generated inner space W and its subspace V satisfy: $(V^\perp)^\perp = V$ and also $\dim(V) + \dim(V^\perp) = \dim(W)$.

Proof: Choose an orthonormal basis X of V and extend it to an orthonormal basis Z of W.
Denote $Y = Z \setminus X$, $X = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k)$, $Y = (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_l)$.

Any $\boldsymbol{u} \in \mathcal{L}(X) = V$ and $\boldsymbol{v} \in \mathcal{L}(Y)$ are orthogonal:

$$\langle \boldsymbol{u} | \boldsymbol{v} \rangle = \Big\langle \sum_{i=1}^{n} a_i \boldsymbol{x}_i \Big| \sum_{j=1}^{n} b_j \boldsymbol{y}_j \Big\rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{b_j} \langle \boldsymbol{x}_i | \boldsymbol{y}_j \rangle = 0$$

as Z is an orthonormal basis. Hence $\mathcal{L}(Y) \subseteq V^\perp$.

Now choose an $\boldsymbol{w} \in V^\perp$ and consider $[\boldsymbol{w}]_Z$. Since Z is orthonormal, the coefficients of $\boldsymbol{w}$ w.r.t. Z are the Fourier coefficients given by the inner product of $\boldsymbol{w}$ and the elements of Z.
Since $\boldsymbol{w} \in V^\perp$, we have $\langle \boldsymbol{w} | \boldsymbol{x}_i \rangle = 0$ for each $\boldsymbol{x}_i \in X$, hence $\boldsymbol{w} \in \mathcal{L}(Y)$, i.e. $V^\perp \subseteq \mathcal{L}(Y)$, and thus $V^\perp = \mathcal{L}(Y)$.

Now: $\dim(V) + \dim(V^\perp) = |X| + |Y| = |Z| = \dim(W)$
and also: $(V^\perp)^\perp = \mathcal{L}(Z \setminus Y) = \mathcal{L}(X) = V$.