Orthogonality

Definition: Vectors $\boldsymbol{u}, \boldsymbol{v}$ from an inner space are *orthogonal* if $\langle \boldsymbol{u} | \boldsymbol{v} \rangle = 0$. We denote orthogonal vectors as $\boldsymbol{u} \perp \boldsymbol{v}$.

Observation: A set of nontrivial pairwise orthogonal vectors is linearly independent.

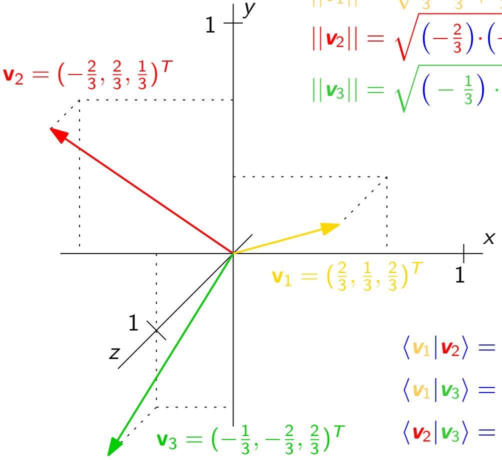
Proof: Assume $\boldsymbol{u}_0, \ldots, \boldsymbol{u}_k$ are orthogonal, but $\boldsymbol{u}_0 = \sum_{i=1}^{k} a_i \boldsymbol{u}_i \neq \mathbf{0}$.
Then:

$$0 \neq \langle \boldsymbol{u}_0 | \boldsymbol{u}_0 \rangle = \left\langle \sum_{i=1}^{k} a_i \boldsymbol{u}_i \middle| \boldsymbol{u}_0 \right\rangle = \sum_{i=1}^{k} a_i \langle \boldsymbol{u}_i | \boldsymbol{u}_0 \rangle = \sum_{i=1}^{k} a_i \cdot 0 = 0$$

Definition: A basis $Z = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ of an inner space V is *orthonormal*, if $\boldsymbol{v}_i \perp \boldsymbol{v}_j$, when $i \neq j$; and $||\boldsymbol{v}_i|| = 1$ for each $\boldsymbol{v}_i \in Z$.

Observe that the matrix whose columns are vectors of an orthonormal basis of $\mathbb{C}^n$ (with standard i.p.) is unitary: $\boldsymbol{A}^H \boldsymbol{A} = \boldsymbol{I}_n$.

Example — an orthonormal basis in $\mathbb{R}^3$



$$||\boldsymbol{v}_1|| = \sqrt{\tfrac{2}{3}\cdot\tfrac{2}{3} + \tfrac{1}{3}\cdot\tfrac{1}{3} + \tfrac{2}{3}\cdot\tfrac{2}{3}} = 1$$

$$||\boldsymbol{v}_2|| = \sqrt{\left(-\tfrac{2}{3}\right)\cdot\left(-\tfrac{2}{3}\right) + \tfrac{2}{3}\cdot\tfrac{2}{3} + \tfrac{1}{3}\cdot\tfrac{1}{3}} = 1$$

$$||\boldsymbol{v}_3|| = \sqrt{\left(-\tfrac{1}{3}\right)\cdot\left(-\tfrac{1}{3}\right) + \left(-\tfrac{2}{3}\right)\cdot\left(-\tfrac{2}{3}\right) + \tfrac{2}{3}\cdot\tfrac{2}{3}} = 1$$

$$\langle \boldsymbol{v}_1 | \boldsymbol{v}_2 \rangle = \tfrac{2}{3}\cdot\left(-\tfrac{2}{3}\right) + \tfrac{1}{3}\cdot\tfrac{2}{3} + \tfrac{2}{3}\cdot\tfrac{1}{3} = 0$$

$$\langle \boldsymbol{v}_1 | \boldsymbol{v}_3 \rangle = \tfrac{2}{3}\cdot\left(-\tfrac{1}{3}\right) + \tfrac{1}{3}\cdot\left(-\tfrac{2}{3}\right) + \tfrac{2}{3}\cdot\tfrac{2}{3} = 0$$

$$\langle \boldsymbol{v}_2 | \boldsymbol{v}_3 \rangle = \left(-\tfrac{2}{3}\right)\cdot\left(-\tfrac{1}{3}\right) + \tfrac{2}{3}\cdot\left(-\tfrac{2}{3}\right) + \tfrac{1}{3}\cdot\tfrac{2}{3} = 0$$

Examples — orthonormal systems of real functions

For the standard inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$ the usual basis $(1, x, x^2)$ *is not* an orthonormal basis of the space of real polynomials of degree at most two on the interval $(0, 1)$:

$$||1|| = \sqrt{\int_0^1 1 \cdot 1\, dx} = \sqrt{[x]_0^1} = 1$$

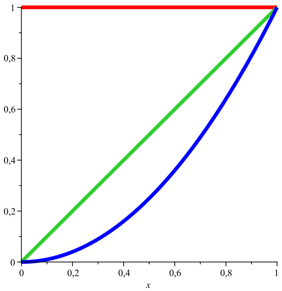
$$||x|| = \sqrt{\int_0^1 x \cdot x\, dx} = \sqrt{[\tfrac{1}{3}x^3]_0^1} = \tfrac{\sqrt{3}}{3} \neq 1$$

$$||x^2|| = \sqrt{\int_0^1 x^2 \cdot x^2\, dx} = \sqrt{[\tfrac{1}{5}x^5]_0^1} = \tfrac{\sqrt{5}}{5} \neq 1$$

$$\langle 1|x\rangle = \int_0^1 1 \cdot x\, dx = [\tfrac{1}{2}x^2]_0^1 = \tfrac{1}{2} \neq 0$$

$$\langle 1|x^2\rangle = \int_0^1 1 \cdot x^2\, dx = [\tfrac{1}{3}x^3]_0^1 = \tfrac{1}{3} \neq 0$$

$$\langle x|x^2\rangle = \int_0^1 x \cdot x^2\, dx = [\tfrac{1}{4}x^4]_0^1 = \tfrac{1}{4} \neq 0$$



It is necessary to take another basis, e.g.
$(1, \sqrt{3}(2x-1), \sqrt{5}(6x^2-6x+1))$:

$$||1|| = \sqrt{\int_0^1 1\cdot 1\, dx} = \sqrt{[x]_0^1} = 1$$

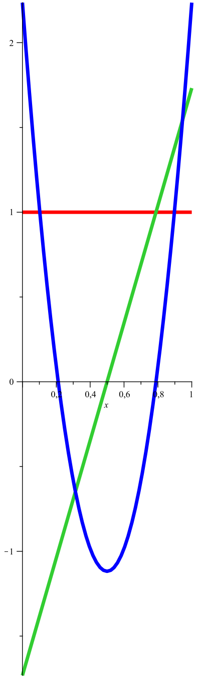
$$||\sqrt{3}(2x-1)|| = \sqrt{\int_0^1 3(4x^2-4x+1)\, dx} =$$
$$= \sqrt{3[\tfrac{4}{3}x^3 - 2x^2 + x]_0^1} = 1$$

$$||\sqrt{5}(6x^2-6x+1)|| =$$
$$= \sqrt{\int_0^1 5(36x^4 - 72x^3 + 48x^2 - 12x + 1)\, dx} =$$
$$= \sqrt{5[\tfrac{36}{5}x^5 - 18x^4 + 16x^3 - 6x^2 + x]_0^1} = 1$$

$$\langle 1|\sqrt{3}(2x-1)\rangle = \int_0^1 \sqrt{3}(2x-1)\, dx = \sqrt{3}[x^2 - x]_0^1 = 0$$

$$\langle 1|\sqrt{5}(6x^2-6x+1)\rangle = \int_0^1 \sqrt{5}(6x^2-6x+1)\, dx =$$
$$= \sqrt{5}[2x^3 - 3x^2 + x]_0^1 = 0$$

$$\langle \sqrt{3}(2x-1)|\sqrt{5}(6x^2-6x+1)\rangle =$$
$$= \int_0^1 \sqrt{15}(12x^3 - 18x^2 + 8x - 1)\, dx =$$
$$= \sqrt{15}[3x^4 - 6x^3 + 4x^2 - x]_0^1 = 0$$



An orthonormal system of periodic functions

Functions $\sin(ix)$ and $\cos(jx)$ are orthogonal on $(-\pi, \pi)$, for $i, j \in \mathbb{N}$.

$$\langle \sin(ix) | \sin(jx) \rangle = \int_{-\pi}^{\pi} \frac{1}{2}[\cos((i-j)x) - \cos((i+j)x)]\, dx =$$
$$= \frac{1}{2}[\frac{1}{i-j} \sin((i-j)x) - \frac{1}{i+j} \sin((i+j)x)]_{-\pi}^{\pi} = 0 \qquad \text{for } i \neq j$$

$$\langle \cos(ix) | \cos(jx) \rangle = \int_{-\pi}^{\pi} \frac{1}{2}[\cos((i-j)x) + \cos((i+j)x)]\, dx =$$
$$= \frac{1}{2}[\frac{1}{i-j} \sin((i-j)x) + \frac{1}{i+j} \sin((i+j)x)]_{-\pi}^{\pi} = 0 \qquad \text{for } i \neq j$$

$$\langle \sin(ix) | \cos(jx) \rangle = \int_{-\pi}^{\pi} \sin(ix) \cos(jx)\, dx = 0$$

We use the fact that $\sin(k\pi) = 0$ for an integer k.
In the last case we integrate an odd function on a symmetric interval.

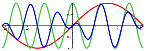
$$|| \sin(ix)||^2 = \int_{-\pi}^{\pi} \frac{1}{2}[1 - \cos(2ix)]\, dx = \frac{1}{2}[x - \frac{1}{2i} \sin(2ix)]_{-\pi}^{\pi} = \pi$$

$$|| \cos(ix)||^2 = \int_{-\pi}^{\pi} \frac{1}{2}[1 + \cos(2ix)]\, dx = \frac{1}{2}[x + \frac{1}{2i} \sin(2ix)]_{-\pi}^{\pi} = \pi$$

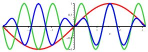
…after a normalization by $\frac{\sqrt{\pi}}{\pi}$ we would get an orthonormal system.

We use formuli:

$\sin\alpha \sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$
$\cos\alpha \cos\beta = \frac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)]$
$\sin\alpha \cos\beta = \frac{1}{2}[\sin(\alpha - \beta) + \sin(\alpha + \beta)]$



$\sin(x) \perp \cos(5x)$



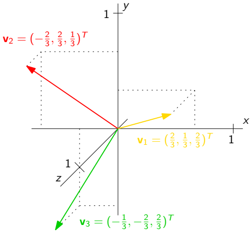
$\sin(x) \perp \sin(5x)$

Properties of an orthonormal basis

Proposition: Let $Z = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$ be an orthonormal basis of an inner space V. Then for any $\boldsymbol{u} \in V$: $\boldsymbol{u} = \langle \boldsymbol{u} | \boldsymbol{v}_1 \rangle \boldsymbol{v}_1 + \dots + \langle \boldsymbol{u} | \boldsymbol{v}_n \rangle \boldsymbol{v}_n$.
The coefficients $\langle \boldsymbol{u} | \boldsymbol{v}_i \rangle$ are called the *Fourier coefficients*.

Proof: If $\boldsymbol{u} = \sum_{i=1}^{n} a_i \boldsymbol{v}_i$, $\langle \boldsymbol{u} | \boldsymbol{v}_j \rangle = \left\langle \sum_{i=1}^{n} a_i \boldsymbol{v}_i \middle| \boldsymbol{v}_j \right\rangle = \sum_{i=1}^{n} a_i \langle \boldsymbol{v}_i | \boldsymbol{v}_j \rangle = a_j$

Coordinates of the vector $\boldsymbol{u} = (3,3,3)^T$ w.r.t. the basis $Z = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ are: $[\boldsymbol{u}]_Z = \Big(\langle \boldsymbol{u} | \boldsymbol{v}_1 \rangle, \langle \boldsymbol{u} | \boldsymbol{v}_2 \rangle, \langle \boldsymbol{u} | \boldsymbol{v}_3 \rangle\Big)^T = (5, 1, -1)^T$



$\langle \boldsymbol{u} | \boldsymbol{v}_1 \rangle = 3 \cdot \frac{2}{3} + 3 \cdot \frac{1}{3} + 3 \cdot \frac{2}{3} = 5$

$\langle \boldsymbol{u} | \boldsymbol{v}_2 \rangle = 3 \cdot \left(-\frac{2}{3}\right) + 3 \cdot \frac{2}{3} + 3 \cdot \frac{1}{3} = 1$

$\langle \boldsymbol{u} | \boldsymbol{v}_3 \rangle = 3 \cdot \left(-\frac{1}{3}\right) + 3 \cdot \left(-\frac{2}{3}\right) + 3 \cdot \frac{2}{3} = -1$

Test: $5 \cdot \boldsymbol{v}_1 + 1 \cdot \boldsymbol{v}_2 + (-1) \cdot \boldsymbol{v}_3 =$

$$= 5 \cdot \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)^T + 1 \cdot \left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)^T + (-1) \cdot \left(-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)^T = (3,3,3)^T = \boldsymbol{u}$$

Properties of an orthonormal basis

Proposition: Let $Z = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be an orthonormal basis of an inner space V. Then for any $\boldsymbol{u} \in V$: $\boldsymbol{u} = \langle \boldsymbol{u} | \boldsymbol{v}_1 \rangle \boldsymbol{v}_1 + \cdots + \langle \boldsymbol{u} | \boldsymbol{v}_n \rangle \boldsymbol{v}_n$. The coefficients $\langle \boldsymbol{u} | \boldsymbol{v}_i \rangle$ are called the *Fourier coefficients*.

Proof: If $\boldsymbol{u} = \sum_{i=1}^{n} a_i \boldsymbol{v}_i$, $\langle \boldsymbol{u} | \boldsymbol{v}_j \rangle = \left\langle \sum_{i=1}^{n} a_i \boldsymbol{v}_i \middle| \boldsymbol{v}_j \right\rangle = \sum_{i=1}^{n} a_i \langle \boldsymbol{v}_i | \boldsymbol{v}_j \rangle = a_j$

Theorem: Let Z be an orthonormal basis of an inner space V. Then for any $\boldsymbol{u}, \boldsymbol{w} \in V$ holds that: $\langle \boldsymbol{u} | \boldsymbol{w} \rangle = [\boldsymbol{w}]_Z^H [\boldsymbol{u}]_Z$.

Proof: We know that $\boldsymbol{u} = \sum_{i=1}^{n} \langle \boldsymbol{u} | \boldsymbol{v}_i \rangle \boldsymbol{v}_i$ and $\boldsymbol{w} = \sum_{j=1}^{n} \langle \boldsymbol{w} | \boldsymbol{v}_j \rangle \boldsymbol{v}_j$.

$$\langle \boldsymbol{u} | \boldsymbol{w} \rangle = \left\langle \sum_{i=1}^{n} \langle \boldsymbol{u} | \boldsymbol{v}_i \rangle \boldsymbol{v}_i \middle| \sum_{j=1}^{n} \langle \boldsymbol{w} | \boldsymbol{v}_j \rangle \boldsymbol{v}_j \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \boldsymbol{u} | \boldsymbol{v}_i \rangle \overline{\langle \boldsymbol{w} | \boldsymbol{v}_j \rangle} \langle \boldsymbol{v}_i | \boldsymbol{v}_j \rangle$$

$$= \sum_{i=1}^{n} \langle \boldsymbol{u} | \boldsymbol{v}_i \rangle \overline{\langle \boldsymbol{w} | \boldsymbol{v}_i \rangle} = [\boldsymbol{w}]_Z^H [\boldsymbol{u}]_Z$$