Inner product

Definition: An *inner product* on a vector space V over $\mathbb{C}$ is a map, that assigns each pair of vectors $\boldsymbol{u}, \boldsymbol{v} \in V$ a scalar $\langle \boldsymbol{u} | \boldsymbol{v} \rangle \in \mathbb{C}$, and that satisfies the following axioms:

- $\forall \boldsymbol{u} \in V : \langle \boldsymbol{u} | \boldsymbol{u} \rangle \in \mathbb{R}_0^+$
- $\forall \boldsymbol{u} \in V : \langle \boldsymbol{u} | \boldsymbol{u} \rangle = 0 \iff \boldsymbol{u} = \boldsymbol{0}$
- $\forall \boldsymbol{u}, \boldsymbol{v} \in V : \langle \boldsymbol{v} | \boldsymbol{u} \rangle = \overline{\langle \boldsymbol{u} | \boldsymbol{v} \rangle}$
- $\forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V : \langle \boldsymbol{u} + \boldsymbol{v} | \boldsymbol{w} \rangle = \langle \boldsymbol{u} | \boldsymbol{w} \rangle + \langle \boldsymbol{v} | \boldsymbol{w} \rangle$
- $\forall \boldsymbol{u} \in V, \forall a \in \mathbb{C} : \langle a\boldsymbol{u} | \boldsymbol{v} \rangle = a \langle \boldsymbol{u} | \boldsymbol{v} \rangle$

(Formally, each inner product is a map $\langle \bullet | \bullet \rangle : V \times V \to \mathbb{C}$.)

A vector space together with its inner product is called *inner space*.

Observation: When extracting from the second argument we get complex conjugate: $\langle \boldsymbol{u} | a\boldsymbol{v} \rangle = \overline{\langle a\boldsymbol{v} | \boldsymbol{u} \rangle} = \overline{a \langle \boldsymbol{v} | \boldsymbol{u} \rangle} = \overline{a}\,\overline{\langle \boldsymbol{v} | \boldsymbol{u} \rangle} = \overline{a} \langle \boldsymbol{u} | \boldsymbol{v} \rangle$

An inner product on V over $\mathbb{R}$ restricts to $\mathbb{R}$, i.e. the codomain of $\langle \bullet | \bullet \rangle$ is $\mathbb{R}$, $\langle \boldsymbol{v} | \boldsymbol{u} \rangle = \langle \boldsymbol{u} | \boldsymbol{v} \rangle$ in the third axiom and $a \in \mathbb{R}$ in the last.

Inner product examples

The *standard inner product* in the arithmetic vector space $\mathbb{C}^n$:

$$\langle \boldsymbol{u} | \boldsymbol{v} \rangle = u_1 \overline{v_1} + \cdots + u_n \overline{v_n} = \sum_{i=1}^{n} u_i \overline{v_i} = \boldsymbol{v}^H \boldsymbol{u}$$

The *standard inner product* in the arithmetic vector space $\mathbb{R}^n$:

$$\langle \boldsymbol{u} | \boldsymbol{v} \rangle = \sum_{i=1}^{n} u_i v_i = \boldsymbol{v}^T \boldsymbol{u}$$

An inner product on $\mathbb{R}^n$ determined by a *regular* matrix $\boldsymbol{A}$:

$$\langle \boldsymbol{u} | \boldsymbol{v} \rangle = \boldsymbol{v}^T \boldsymbol{A}^T \boldsymbol{A} \boldsymbol{u}$$

e.g. for $V = \mathbb{R}^2$ and $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ we have:

$$\langle \boldsymbol{u} | \boldsymbol{v} \rangle = (v_1, v_2) \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u_1 v_1 + 2 u_1 v_2 + 2 u_2 v_1 + 5 u_2 v_2$$

An inner product on the vector space of real continuous integrable functions on the interval $[a, b]$: $\langle f | g \rangle = \int_a^b f(x) g(x) dx$

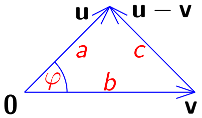
Norm

Definition: Let V be an inner space (over $\mathbb{C}$ or $\mathbb{R}$).
The *norm based upon the inner product*
is the map $||\bullet|| : V \to \mathbb{R}$ defined as $||\boldsymbol{u}|| = \sqrt{\langle \boldsymbol{u}|\boldsymbol{u}\rangle}$.

Geometric interpretation in Euclidean inner spaces:
$||\boldsymbol{u}||$ ... the length of $\boldsymbol{u}$
$||\boldsymbol{u} - \boldsymbol{v}||$ ... the distance between points $\boldsymbol{u}$ and $\boldsymbol{v}$
$\langle \boldsymbol{u}|\boldsymbol{v}\rangle$ ... is related to the "angle" between $\boldsymbol{u}$ and $\boldsymbol{v}$
and also to the length of $\boldsymbol{u}$ and $\boldsymbol{v}$. More precisely:

Observation: The standard inner product on $\mathbb{R}^n$ and the associated norm satisfy: $\langle \boldsymbol{u}|\boldsymbol{v}\rangle = ||\boldsymbol{u}|| \cdot ||\boldsymbol{v}|| \cos\varphi$, where φ is the angle between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ (seen as segments stemming from $\mathbf{0}$).

Proof:



By the law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\varphi$.
Substitute $a = ||\boldsymbol{u}||$, $b = ||\boldsymbol{v}||$, $c = ||\boldsymbol{u} - \boldsymbol{v}||$.
$\langle \boldsymbol{u}-\boldsymbol{v}|\boldsymbol{u}-\boldsymbol{v}\rangle = \langle \boldsymbol{u}|\boldsymbol{u}\rangle + \langle \boldsymbol{v}|\boldsymbol{v}\rangle - 2||\boldsymbol{u}|| \cdot ||\boldsymbol{v}|| \cos\varphi$
$\langle \boldsymbol{u}-\boldsymbol{v}|\boldsymbol{u}-\boldsymbol{v}\rangle = \langle \boldsymbol{u}|\boldsymbol{u}\rangle + \langle \boldsymbol{v}|\boldsymbol{v}\rangle - \langle \boldsymbol{u}|\boldsymbol{v}\rangle - \langle \boldsymbol{v}|\boldsymbol{u}\rangle$

Cauchy-Schwarz inequality

Theorem: The inner product of any two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ from any inner space over $\mathbb{C}$ satisfies: $|\langle \boldsymbol{u}|\boldsymbol{v}\rangle| \leq \sqrt{\langle \boldsymbol{u}|\boldsymbol{u}\rangle\langle \boldsymbol{v}|\boldsymbol{v}\rangle}$.

In other words, w.r.t. the associated norm: $|\langle \boldsymbol{u}|\boldsymbol{v}\rangle| \leq ||\boldsymbol{u}|| \cdot ||\boldsymbol{v}||$.

Proof: When $\boldsymbol{u} = \mathbf{0}$ or $\boldsymbol{v} = \mathbf{0}$, we get $0 \leq 0$.
For any $a \in \mathbb{C}$ it holds that $||\boldsymbol{u} + a\boldsymbol{v}||^2 \geq 0$, but also:
$||\boldsymbol{u} + a\boldsymbol{v}||^2 = \langle \boldsymbol{u} + a\boldsymbol{v}|\boldsymbol{u} + a\boldsymbol{v}\rangle = \langle \boldsymbol{u}|\boldsymbol{u}\rangle + a\langle \boldsymbol{v}|\boldsymbol{u}\rangle + \overline{a}\langle \boldsymbol{u}|\boldsymbol{v}\rangle + a\overline{a}\langle \boldsymbol{v}|\boldsymbol{v}\rangle$

Choose $a = -\frac{\langle \boldsymbol{u}|\boldsymbol{v}\rangle}{\langle \boldsymbol{v}|\boldsymbol{v}\rangle}$ to cancel the last two terms. We get:

$$\begin{array}{rcll} 0 & \leq & \langle \boldsymbol{u}|\boldsymbol{u}\rangle - \frac{\langle \boldsymbol{u}|\boldsymbol{v}\rangle}{\langle \boldsymbol{v}|\boldsymbol{v}\rangle}\langle \boldsymbol{v}|\boldsymbol{u}\rangle & \ldots \cdot \langle \boldsymbol{v}|\boldsymbol{v}\rangle > 0 \\ \langle \boldsymbol{u}|\boldsymbol{v}\rangle\langle \boldsymbol{v}|\boldsymbol{u}\rangle & \leq & \langle \boldsymbol{u}|\boldsymbol{u}\rangle\langle \boldsymbol{v}|\boldsymbol{v}\rangle & \ldots \text{on } \mathbb{C} \text{ holds } a\overline{a} = |a|^2 \\ |\langle \boldsymbol{u}|\boldsymbol{v}\rangle|^2 & \leq & ||\boldsymbol{u}||^2 \cdot ||\boldsymbol{v}||^2 & \ldots \text{take square root} \\ |\langle \boldsymbol{u}|\boldsymbol{v}\rangle| & \leq & ||\boldsymbol{u}|| \cdot ||\boldsymbol{v}|| & \end{array}$$

Consequences

Theorem: (AM-RMS inequality)

For any $\boldsymbol{u} \in \mathbb{R}^n$: $\frac{1}{n}\sum_{i=1}^{n} u_i \leq \sqrt{\frac{1}{n}\sum_{i=1}^{n} u_i^2}$

Proof: Choose $\boldsymbol{v} = (1, 1, \ldots, 1)^T$ and apply Cauchy-Schwarz inequality on the standard inner product:

$$\sum_{i=1}^{n} u_i = \langle \boldsymbol{u}|\boldsymbol{v}\rangle \leq |\langle \boldsymbol{u}|\boldsymbol{v}\rangle| \leq ||\boldsymbol{u}|| \cdot ||\boldsymbol{v}|| = \sqrt{\sum_{i=1}^{n} u_i^2} \cdot \sqrt{n}$$

Proposition: Any norm based upon an inner product satisfies the triangle inequality: $||\boldsymbol{u}+\boldsymbol{v}|| \leq ||\boldsymbol{u}|| + ||\boldsymbol{v}||$

Proof:

$$||\boldsymbol{u}+\boldsymbol{v}|| = \sqrt{\langle \boldsymbol{u}+\boldsymbol{v}|\boldsymbol{u}+\boldsymbol{v}\rangle} = \sqrt{\langle \boldsymbol{u}|\boldsymbol{u}\rangle + \langle \boldsymbol{v}|\boldsymbol{u}\rangle + \langle \boldsymbol{u}|\boldsymbol{v}\rangle + \langle \boldsymbol{v}|\boldsymbol{v}\rangle} \leq$$

$$\sqrt{||\boldsymbol{u}||^2 + 2|\langle \boldsymbol{u}|\boldsymbol{v}\rangle| + ||\boldsymbol{v}||^2} \leq \sqrt{||\boldsymbol{u}||^2 + 2||\boldsymbol{u}|| \cdot ||\boldsymbol{v}|| + ||\boldsymbol{v}||^2} = ||\boldsymbol{u}|| + ||\boldsymbol{v}||$$