A geometric problem

Problem: Can a rectangle with irrational ratio of the length of its sides be partitioned into finitely many squares?

Note: For a rational ratio $p : q$ we may use pq squares 1×1.

Theorem: For an irrational ratio no such partition exists.

Proof: Let the rectangle R has side lengths $1 : x$, where $x \in \mathbb{R} \setminus \mathbb{Q}$.

$\mathbb{R}$ is a vector space over $\mathbb{Q}$. Here 1 and x are linearly independent.

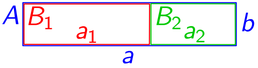
Choose any linear map $f : \mathbb{R} \to \mathbb{R}$, where $f(1) = 1$ and $f(x) = -1$.

Such map f exists as the image of a part of a basis is prescribed.

For any rectangle A of sides a, b define "area" as $v(A) = f(a)f(b)$.

If we iteratively cut A into $B_1, \ldots, B_k$ then $v(A) = \sum_{i=1}^{k} v(B_i)$.

... cutting A into B_1 and B_2 yields $v(A) = v(B_1) + v(B_2)$.



$f(a)f(b) = f(a_1 + a_2)f(b) = (f(a_1) + f(a_2))f(b) = f(a_1)f(b) + f(a_2)f(b)$

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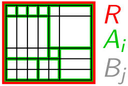
If we iteratively cut A into $B_1, \ldots, B_k$ then $v(A) = \sum_{i=1}^{k} v(B_i)$.

If R could partitioned into squares $A_1, \ldots, A_k$ of side lengths $a_1, \ldots, a_k$, then we refine the partition along their sides to rectangles $B_1, \ldots, B_l$ and get a contradiction.



$$-1 = f(1)f(x) = v(R) = \sum_{j=1}^{l} v(B_j) = \sum_{i=1}^{k} v(A_i) = \sum_{i=1}^{k} f(a_i)^2 \geq 0$$