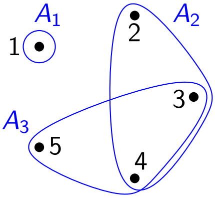
Constrained set systems

Problem: How many subsets may an n-element set contain, if each subset has odd size, but the intersection of each pair of distinct subsets has even size.

Formally:
$\max k : \exists A_1, \ldots, A_k \subseteq \{1, \ldots, n\} \ \forall i \neq j : 2 \nmid |A_i| \wedge 2 \mid |A_i \cap A_j|$

Example: $n = 5, k = 3$

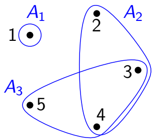


Theorem: It holds that $k \leq n$, i.e. under the precribed constraints, at most n sets $A_1, \ldots, A_n$ exist (e.g. of size one).

Proof that $k \le n$

We construct the so called *incidence matrix* $\boldsymbol{M} \in \mathbb{Z}_2^{k \times n}$:

$$m_{ij} = \begin{cases} 1 & \text{if } j \in A_i \\ 0 & \text{if } j \notin A_i \end{cases}.$$

For 

we get:

$$\boldsymbol{M} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Observe that $\boldsymbol{M}\boldsymbol{M}^{\mathsf{T}} = \mathbf{I}_k$, because over $\mathbb{Z}_2$ we have:

$$(\boldsymbol{M}\boldsymbol{M}^{\mathsf{T}})_{ij} = \begin{cases} 1 & \text{if } i = j\text{, as this is the } \textit{parity} \text{ of } |A_i|, \\ 0 & \text{if } i \neq j\text{, as this is the } \textit{parity} \text{ of } |A_i \cap A_j|. \end{cases}$$

Now $k = \operatorname{rank}(\mathbf{I}_k) = \operatorname{rank}(\boldsymbol{M}\boldsymbol{M}^{\mathsf{T}}) \le \operatorname{rank}(\boldsymbol{M}) \le n$

Note: If we choose $\boldsymbol{M} \in \mathbb{R}^{k \times n}$, then $(\boldsymbol{M}\boldsymbol{M}^{\mathsf{T}})_{ij} = |A_i \cap A_j|$.