Spaces determined by a matrix

Definition: The *kernel* of $\boldsymbol{A} \in \mathbb{K}^{m \times n}$ is $\ker(\boldsymbol{A}) = \{\boldsymbol{x} \in \mathbb{K}^n : \boldsymbol{A}\boldsymbol{x} = \mathbf{0}\}$,
the *row space* $\mathcal{R}(\boldsymbol{A}) \subseteq \mathbb{K}^n$ is the linear hull of the rows of $\boldsymbol{A}$,
the *column space* $\mathcal{C}(\boldsymbol{A}) \subseteq \mathbb{K}^m$ is the linear hull of the columns of $\boldsymbol{A}$.

Example: For the matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 \end{pmatrix} \in \mathbb{Z}_3^{3\times 4}$ we get

The row space:

$$\mathcal{R}(\boldsymbol{A}) = \left\{ \begin{matrix} (0,0,0,0)^T, & (1,2,0,1)^T, & (2,1,0,2)^T, \\ (2,0,2,1)^T, & (0,2,2,2)^T, & (1,1,2,0)^T, \\ (1,0,1,2)^T, & (2,2,1,0)^T, & (0,1,1,1)^T \end{matrix} \right\} \subseteq \mathbb{Z}_3^4$$

The column space:

$$\mathcal{C}(\boldsymbol{A}) = \left\{ \begin{matrix} (0,0,0)^T, & (1,2,1)^T, & (2,1,2)^T, \\ (2,0,1)^T, & (0,2,2)^T, & (1,1,0)^T, \\ (1,0,2)^T, & (2,2,0)^T, & (0,1,1)^T \end{matrix} \right\} \subseteq \mathbb{Z}_3^3$$

Lemma: If $\boldsymbol{A}' = \boldsymbol{B}\boldsymbol{A}$ then $\dim(\mathcal{C}(\boldsymbol{A}')) \leq \dim(\mathcal{C}(\boldsymbol{A}))$

Example: For $\boldsymbol{A}' = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \boldsymbol{B}\boldsymbol{A}$ we get $\dim(\mathcal{C}(\boldsymbol{A}')) = 1 \leq 2 = \dim(\mathcal{C}(\boldsymbol{A}))$. The matrix $\boldsymbol{B}$ is intentionally chosen singular to get $\dim(\mathcal{C}(\boldsymbol{A}'))$ strictly smaller than $\dim(\mathcal{C}(\boldsymbol{A}))$.

Proof: Denote by $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ the columns of $\boldsymbol{A}$.
The columns $\boldsymbol{u}_1', \ldots, \boldsymbol{u}_n'$ of $\boldsymbol{A}'$ satisfy $\boldsymbol{u}_i' = \boldsymbol{B}\boldsymbol{u}_i$.
When $\boldsymbol{w}' \in \mathcal{C}(\boldsymbol{A}')$ then $\boldsymbol{w}' = \sum_{i=1}^{n} a_i \boldsymbol{u}_i'$ for some $a_1, \ldots, a_n \in \mathbb{K}$.
Thus $\boldsymbol{w}' = \sum_{i=1}^{n} a_i \boldsymbol{B}\boldsymbol{u}_i = \boldsymbol{B} \sum_{i=1}^{n} a_i \boldsymbol{u}_i = \boldsymbol{B}\boldsymbol{w}$ for $\boldsymbol{w} = \sum_{i=1}^{n} a_i \boldsymbol{u}_i \in \mathcal{C}(\boldsymbol{A})$.
W.l.o.g. $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d$ form a basis of $\mathcal{C}(\boldsymbol{A})$, $d = \dim(\mathcal{C}(\boldsymbol{A}))$, i.e.
$\boldsymbol{w} = \sum_{i=1}^{d} b_i \boldsymbol{u}_i$ for some $b_1, \ldots, b_d \in \mathbb{K}$.
Since $\boldsymbol{w}' = \boldsymbol{B}\boldsymbol{w} = \boldsymbol{B} \sum_{i=1}^{d} b_i \boldsymbol{u}_i = \sum_{i=1}^{d} b_i \boldsymbol{B}\boldsymbol{u}_i = \sum_{i=1}^{d} b_i \boldsymbol{u}_i'$ we get that $\boldsymbol{u}_1', \ldots, \boldsymbol{u}_d'$ generate $\mathcal{C}(\boldsymbol{A}')$. Hence $\dim(\mathcal{C}(\boldsymbol{A}')) \leq d = \dim(\mathcal{C}(\boldsymbol{A}))$.

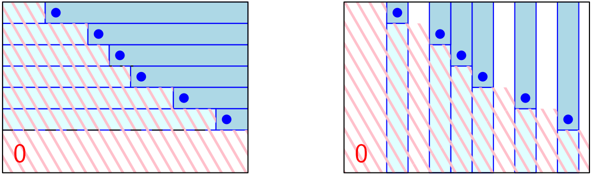
Both spaces have the same dimension

Theorem: Any $\boldsymbol{A} \in \mathbb{K}^{m\times n}$ satisfies: $\dim(\mathcal{R}(\boldsymbol{A})) = \dim(\mathcal{C}(\boldsymbol{A}))$.

Proof: Let $\boldsymbol{A} \sim\sim \boldsymbol{A}'$ in reduced echelon form, i.e. there is a *regular* $\boldsymbol{R}$ s.t. $\boldsymbol{A}' = \boldsymbol{R}\boldsymbol{A}$. By the lemma $\dim(\mathcal{C}(\boldsymbol{A}')) \leq \dim(\mathcal{C}(\boldsymbol{A}))$.

From $\boldsymbol{A} = \boldsymbol{R}^{-1}\boldsymbol{A}'$ we get $\dim(\mathcal{C}(\boldsymbol{A}')) \geq \dim(\mathcal{C}(\boldsymbol{A}))$ and indeed $=$.

For matrices $\boldsymbol{A}'$ in the echelon form the theorem holds directly:
$\dim(\mathcal{R}(\boldsymbol{A}')) = \#$ of pivots $= \operatorname{rank}(\boldsymbol{A}') = \dim(\mathcal{C}(\boldsymbol{A}'))$



Since $\mathcal{R}(\boldsymbol{A}) = \mathcal{R}(\boldsymbol{A}')$, we get
$\dim(\mathcal{R}(\boldsymbol{A})) = \dim(\mathcal{R}(\boldsymbol{A}')) = \dim(\mathcal{C}(\boldsymbol{A}')) = \dim(\mathcal{C}(\boldsymbol{A}))$.

Example

Use Gauss-Jordan elimination:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 \end{pmatrix} \sim\sim \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \boldsymbol{A}'$$

$\mathcal{R}(\boldsymbol{A}) = \mathcal{R}(\boldsymbol{A}')$ yields $\dim(\mathcal{R}(\boldsymbol{A})) = \dim(\mathcal{R}(\boldsymbol{A}')) = \operatorname{rank}(\boldsymbol{A}) = 2$.

From $\boldsymbol{A}' = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 2 & 0 \\ 1 & 2 & 1 \end{pmatrix} \boldsymbol{A}$ we get $\dim(\mathcal{C}(\boldsymbol{A}')) \leq \dim(\mathcal{C}(\boldsymbol{A}))$.

The matrix of the transformation is regular — it has an inverse, so

from $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \boldsymbol{A}'$ we get $\dim(\mathcal{C}(\boldsymbol{A})) \leq \dim(\mathcal{C}(\boldsymbol{A}'))$.

The space $\mathcal{C}(\boldsymbol{A}')$ is generated by the *columns with pivots*, here by the first two vectors of the standard basis. Hence $\dim(\mathcal{C}(\boldsymbol{A}')) = 2$. In particular $\mathcal{C}(\boldsymbol{A}') = \{(p_1, p_2, 0)^T, p_1, p_2 \in \mathbb{Z}_3\}$.

Consequences

- $\operatorname{rank}(\boldsymbol{A}^T) = \operatorname{rank}(\boldsymbol{A})$
- For any $\boldsymbol{A} \in \mathbb{K}^{m \times n}$ and regular $\boldsymbol{R} \in \mathbb{K}^{m \times m}$, $\boldsymbol{R}' \in \mathbb{K}^{n \times n}$:
  $\operatorname{rank}(\boldsymbol{RA}) = \operatorname{rank}(\boldsymbol{AR}') = \operatorname{rank}(\boldsymbol{A})$
- $\mathcal{R}(\boldsymbol{BA}) \subseteq \mathcal{R}(\boldsymbol{A}), \quad \mathcal{C}(\boldsymbol{BA}) \subseteq \mathcal{C}(\boldsymbol{B})$
- $\operatorname{rank}(\boldsymbol{BA}) \leq \min\{\operatorname{rank}(\boldsymbol{A}), \operatorname{rank}(\boldsymbol{B})\}$

Theorem: For any $\boldsymbol{A} \in \mathbb{K}^{m \times n} : \dim(\ker(\boldsymbol{A})) + \operatorname{rank}(\boldsymbol{A}) = n$

Proof: Let $d = n - \operatorname{rank}(\boldsymbol{A})$ be the number of free variables and $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d$ solutions of $\boldsymbol{Ax} = \mathbf{0}$ obtained by backward substitution.

These solutions are linearly independent as for each i,
$\boldsymbol{x}_i$ is the only one among $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d$ which has
the cordinate corresponding to the i-th free variable nonzero.

Vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d$ form a basis of $\ker(\boldsymbol{A})$
and $\dim(\ker(\boldsymbol{A})) = d = n - \operatorname{rank}(\boldsymbol{A})$.