Solution verification for $\boldsymbol{Ax} = \boldsymbol{b}$

Substitute $\boldsymbol{x}$ *including parameters into the original system* $\boldsymbol{Ax} = \boldsymbol{b}$*, ie., verify* $\boldsymbol{x}^0$ *for* $\boldsymbol{Ax} = \boldsymbol{b}$ *and also* $\overline{\boldsymbol{x}}^1, \dots, \overline{\boldsymbol{x}}^{n-r}$ *for* $\boldsymbol{A}\overline{\boldsymbol{x}} = \mathbf{0}$*.*

This test does not verify the completeness of the solution set, because it may happen that we add a new condition due to an error in Gaussian elimination. Then we get only a subset of all solutions.

For simplicity, we only consider homogeneous systems $\boldsymbol{Ax} = \mathbf{0}$, i.e. only the matrix $\boldsymbol{A}$. For nonhomogeneous systems, it will be necessary to take the augmented matrix $(\boldsymbol{A}|\boldsymbol{b})$ instead.

The correctness of Gaussian elimination can be verified, eg. by its reversal, i.e. by transforming the matrix $\boldsymbol{A}'$ in the echelon form to the original matrix $\boldsymbol{A}$ by elementary transforms.

This can be done by first adding the identity matrix to the matrix $\boldsymbol{A}$ and eliminating both together $(\boldsymbol{A}|\mathbf{I}) \sim\sim (\boldsymbol{A}'|\boldsymbol{C})$.
Then we move the "control" block $\boldsymbol{C}$ in front of $\boldsymbol{A}'$ and by Gauss-Jordan elimination we reverse the process $(\boldsymbol{C}|\boldsymbol{A}') \sim\sim (\mathbf{I}|\boldsymbol{A})$.

Example

Gaussian elimination with the identity matrix added:

$$(\boldsymbol{A}|\mathbf{I}_4) = \left(\begin{array}{ccccc|cccc} 1 & 4 & 3 & 2 & 1 & 1 & 0 & 0 & 0 \\ 2 & 8 & 4 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 6 & 9 & 0 & 0 & 1 & 0 \\ 2 & 8 & 7 & 6 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ -2\mathrm{I} \\ \\ -2\mathrm{I} \end{array} \sim\sim \left(\begin{array}{ccccc|cccc} 1 & 4 & 3 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & -4 & -2 & -2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 6 & 9 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & -2 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} \\ \mathrm{IV} \\ -3\mathrm{IV} \\ \mathrm{II}+2\mathrm{IV} \end{array}$$

$$\sim\sim \left(\begin{array}{ccccc|cccc} 1 & 4 & 3 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 6 & 6 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & -6 & 1 & 0 & 2 \end{array}\right) = (\boldsymbol{A}'|\boldsymbol{C})$$

Transformation in the opposite direction: $(\boldsymbol{C}|\boldsymbol{A}') =$

$$= \left(\begin{array}{cccc|ccccc} 1 & 0 & 0 & 0 & 1 & 4 & 3 & 2 & 1 \\ -2 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 1 \\ 6 & 0 & 1 & -3 & 0 & 0 & 0 & 0 & 6 \\ -6 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} \\ +2\mathrm{I} \\ -6\mathrm{I} \\ +6\mathrm{I} \end{array} \sim\sim \left(\begin{array}{cccc|ccccc} 1 & 0 & 0 & 0 & 1 & 4 & 3 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 8 & 7 & 6 & 3 \\ 0 & 0 & 1 & -3 & -6 & -24 & -18 & -12 & 0 \\ 0 & 1 & 0 & 2 & 6 & 24 & 18 & 12 & 6 \end{array}\right) \begin{array}{c} \mathrm{IV}-2\mathrm{II} \\ +3\mathrm{II} \\ \mathrm{II} \end{array}$$

$$\sim\sim \left(\begin{array}{cccc|ccccc} 1 & 0 & 0 & 0 & 1 & 4 & 3 & 2 & 1 \\ 0 & 1 & 0 & 0 & 2 & 8 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 3 & 6 & 9 \\ 0 & 0 & 0 & 1 & 2 & 8 & 7 & 6 & 3 \end{array}\right) = (\mathbf{I}_4|\boldsymbol{A})$$

Solution verification for $\boldsymbol{Ax} = \boldsymbol{b}$

Another way to test the completeness of the solution is to verify that the rank of $\boldsymbol{A}$ was not calculated *higher* than it actually is.

The columns of the matrix $\boldsymbol{A}$ corresponding to the leading variables should be linearly independent. From these columns, we compose the matrix $\boldsymbol{B}$ and independently determine its rank.

If the rank matches the number of columns in $\boldsymbol{B}$, then these columns are linearly independent.

To avoid the same sequence of elementary transformations as in the Gaussian elimination of $\boldsymbol{A}$, we can either shuffle these columns, or we can compute the rank of $\boldsymbol{B}^\mathsf{T}$ instead.

Since $\boldsymbol{B}^\mathsf{T}$ has at least as many columns as rows, Gaussian elimination of $\boldsymbol{B}^\mathsf{T}$ can be faster than the elimination of $\boldsymbol{B}$.

Example

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 & 3 & 2 & 1 \\ 2 & 8 & 4 & 0 & 0 \\ 0 & 0 & 3 & 6 & 9 \\ 2 & 8 & 7 & 6 & 3 \end{pmatrix} \sim\sim \begin{pmatrix} 1 & 4 & 3 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \boldsymbol{A}'$$

From the columns of $\boldsymbol{A}$ corresponding to the leading variables, we compose the matrix $\boldsymbol{B}$ and independently determine its rank, e.g. by using $\operatorname{rank}(\boldsymbol{B}) = \operatorname{rank}(\boldsymbol{B}^{\mathsf{T}})$.

$$\boldsymbol{B} = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 4 & 0 \\ 0 & 3 & 9 \\ 2 & 7 & 3 \end{pmatrix}$$

$$\boldsymbol{B}^{\mathsf{T}} = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 3 & 4 & 3 & 7 \\ 1 & 0 & 9 & 3 \end{pmatrix} \begin{array}{c} \\ \sim -3\mathrm{I} \sim \\ -\mathrm{I} \end{array} \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & -2 & 3 & 1 \\ 0 & -2 & 9 & 1 \end{pmatrix} \begin{array}{c} \\ \sim \\ -\mathrm{II} \end{array} \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 6 & 0 \end{pmatrix}$$

As $\operatorname{rank}(\boldsymbol{B}^{\mathsf{T}}) = \operatorname{rank}(\boldsymbol{A})$, the rank of $\boldsymbol{A}$ was determined correctly.

Solution verification for $\boldsymbol{Ax} = \boldsymbol{b}$

A combination of both ways is the property test from the second:
"the rank of $\boldsymbol{A}$ was not calculated higher than it actually is"
with help of the control matrix $\boldsymbol{C}$ from the first and the product.

Just verify that $\boldsymbol{A}' = \boldsymbol{CA}$,
because then it holds that:
$\operatorname{rank}(\boldsymbol{A}') = \operatorname{rank}(\boldsymbol{CA}) \leq \operatorname{rank}(\boldsymbol{A})$.

Example:

$$\begin{array}{rrrr|rrrrr}
 & & & & 1 & 4 & 3 & 2 & 1 \\
 & & & & 2 & 8 & 4 & 0 & 0 \\
 & & & & 0 & 0 & 3 & 6 & 9 \\
 & & & & 2 & 8 & 7 & 6 & 3 \\
\hline
1 & 0 & 0 & 0 & 1 & 4 & 3 & 2 & 1 \\
-2 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 1 \\
6 & 0 & 1 & -3 & 0 & 0 & 0 & 0 & 6 \\
-6 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0
\end{array}$$

Out of the three ways, this is perhaps the computationally simplest, because instead of the second elimination, only a product suffices. Still, we shall not forget to test the solution of the system.

Why are these methods correct?

Later we will show that:

- Elementary transforms are products with suitable matrices.
- The elimination of $\boldsymbol{C}$ on $\mathbf{I}$ corresponds to the product with the matrix $\boldsymbol{C}^{-1}$ from the left.
- The product with $\boldsymbol{C}^{-1}$ performs the reverse process compared to the product with $\boldsymbol{C}$.

- The rank of a matrix is the number of its linearly independent columns. (We will also define linear independence.)
- $\operatorname{rank}(\boldsymbol{B}) = \operatorname{rank}(\boldsymbol{B}^{\mathsf{T}})$

- The rank cannot increase in the matrix product, i.e. $\operatorname{rank}(\boldsymbol{CA}) \leq \operatorname{rank}(\boldsymbol{A})$.