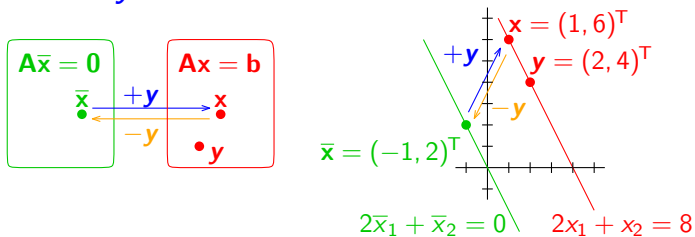


Homogeneous and non-homogeneous systems

Notation: Vectors of the same length are added by components. We multiply a vector by a real number also by components.

Observation: If \mathbf{x} and \mathbf{y} are two solutions of $\mathbf{Ax} = \mathbf{b}$ then $\bar{\mathbf{x}} = \mathbf{x} - \mathbf{y}$ is a solution of $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$.



Proof: $\mathbf{A}\bar{\mathbf{x}} = \mathbf{A}(\mathbf{x} - \mathbf{y}) \stackrel{*}{=} \mathbf{Ax} - \mathbf{Ay} = \mathbf{b} - \mathbf{b} = \mathbf{0}$.

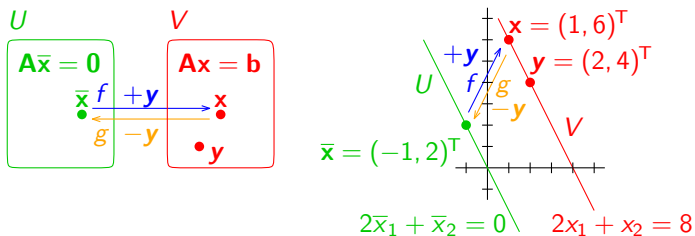
*: $a_{i1}(x_1 - y_1) + \dots + a_{in}(x_n - y_n) = (a_{i1}x_1 + \dots + a_{in}x_n) - (a_{i1}y_1 + \dots + a_{in}y_n)$

Example: $2(-1) + 2 = 2(1 - 2) + (6 - 4) \stackrel{*}{=} (2 \cdot 1 + 6) - (2 \cdot 2 + 4) = 8 - 8 = 0$

Observation: If \mathbf{y} is a solution of $\mathbf{Ax} = \mathbf{b}$ and $\bar{\mathbf{x}}$ is a solution of $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$ then $\mathbf{x} = \bar{\mathbf{x}} + \mathbf{y}$ is a solution of $\mathbf{Ax} = \mathbf{b}$.

Proof: Analogously: $\mathbf{Ax} = \mathbf{A}(\bar{\mathbf{x}} + \mathbf{y}) = \mathbf{A}\bar{\mathbf{x}} + \mathbf{Ay} = \mathbf{b} + \mathbf{0} = \mathbf{b}$.

Homogeneous and non-homogeneous systems



Theorem: Let \mathbf{y} satisfy $\mathbf{A}\mathbf{y} = \mathbf{b}$. Then the map $\bar{\mathbf{x}} \rightarrow \bar{\mathbf{x}} + \mathbf{y}$ is a bijection between the sets $\{\bar{\mathbf{x}}: \mathbf{A}\bar{\mathbf{x}} = \mathbf{0}\}$ and $\{\mathbf{x}: \mathbf{A}\mathbf{x} = \mathbf{b}\}$.

Proof: Denote $U = \{\bar{\mathbf{x}}: \mathbf{A}\bar{\mathbf{x}} = \mathbf{0}\}$, $V = \{\mathbf{x}: \mathbf{A}\mathbf{x} = \mathbf{b}\}$,
 $f: U \rightarrow V$ by $f(\bar{\mathbf{x}}) = \bar{\mathbf{x}} + \mathbf{y}$, and $g: V \rightarrow U$ by $g(\mathbf{x}) = \mathbf{x} - \mathbf{y}$.

$f \circ g$ is the identity on $V \implies f$ is surjective
 $g \circ f$ is the identity on $U \implies f$ is injective
 $\left. \vphantom{\begin{matrix} f \circ g \\ g \circ f \end{matrix}} \right\} \implies f$ is bijective.

Solutions of homogeneous systems $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$

Theorem: For a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ of rank r , all solutions of $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$ can be written as $\bar{\mathbf{x}} = p_1\bar{\mathbf{x}}_1 + p_2\bar{\mathbf{x}}_2 + \cdots + p_{n-r}\bar{\mathbf{x}}_{n-r}$, where

- ▶ p_1, \dots, p_{n-r} are arbitrary real parameters and
- ▶ $\bar{\mathbf{x}}_1, \dots, \bar{\mathbf{x}}_{n-r}$ are suitable solutions of $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$.

The system has only the trivial solution $\bar{\mathbf{x}} = \mathbf{0}$ iff $\text{rank}(\mathbf{A}) = n$.

Example:

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 3 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 6 \end{pmatrix} \quad \begin{array}{l} \bar{x}_1 = -4\bar{x}_2 - 3\bar{x}_3 - 2\bar{x}_4 - \bar{x}_5 = -4\bar{x}_2 + 4\bar{x}_4 \\ \bar{x}_3 = -2\bar{x}_4 - \bar{x}_5 = -2\bar{x}_4 \\ \bar{x}_5 = 0 \end{array}$$

When we rename \bar{x}_2 and \bar{x}_4 by parameters p_1 and p_2 , we get:

$$\begin{array}{l} \bar{x}_1 = -4\bar{x}_2 + 4\bar{x}_4 = -4p_1 + 4p_2 \\ \bar{x}_2 = p_1 \\ \bar{x}_3 = -2\bar{x}_4 = -2p_2 \\ \bar{x}_4 = p_2 \\ \bar{x}_5 = 0 \end{array} \quad \text{that is } \bar{\mathbf{x}} = p_1 \bar{\mathbf{x}}_1 + p_2 \bar{\mathbf{x}}_2 = p_1 \begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + p_2 \begin{pmatrix} 4 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}.$$

Solutions of homogeneous systems $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$

Theorem: For a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ of rank r , all solutions of $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$ can be written as $\bar{\mathbf{x}} = p_1\bar{\mathbf{x}}_1 + p_2\bar{\mathbf{x}}_2 + \cdots + p_{n-r}\bar{\mathbf{x}}_{n-r}$, where

- ▶ p_1, \dots, p_{n-r} are arbitrary real parameters and
- ▶ $\bar{\mathbf{x}}_1, \dots, \bar{\mathbf{x}}_{n-r}$ are suitable solutions of $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$.

The system has only the trivial solution $\bar{\mathbf{x}} = \mathbf{0}$ iff $\text{rank}(\mathbf{A}) = n$.

Proof: Rename the free variables as *parameters* p_1, \dots, p_{n-r} .

By backward substitution we express each solution component as a linear function of free variables with *coefficients* $q_{1,1}, \dots, q_{n,n-r}$:

$$\begin{aligned}\bar{x}_1 &= q_{1,1}p_1 + \cdots + q_{1,n-r}p_{n-r} \\ &\vdots \\ \bar{x}_n &= q_{n,1}p_1 + \cdots + q_{n,n-r}p_{n-r}\end{aligned}$$

Choose $\bar{\mathbf{x}}_1 = (q_{1,1}, \dots, q_{n,1})^T, \dots, \bar{\mathbf{x}}_{n-r} = (q_{1,n-r}, \dots, q_{n,n-r})^T$.

These solve $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$ as any such $\bar{\mathbf{x}}_i$ comes from: $p_j = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$

If $\text{rank}(\mathbf{A}) = n$, all variables are leading, and $\mathbf{0}$ is the only solution.

Solutions of homogeneous systems $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$

Theorem: For a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ of rank r , all solutions of $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$ can be written as $\bar{\mathbf{x}} = p_1\bar{\mathbf{x}}_1 + p_2\bar{\mathbf{x}}_2 + \cdots + p_{n-r}\bar{\mathbf{x}}_{n-r}$, where

- ▶ p_1, \dots, p_{n-r} are arbitrary real parameters and
- ▶ $\bar{\mathbf{x}}_1, \dots, \bar{\mathbf{x}}_{n-r}$ are suitable solutions of $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$.

The system has only the trivial solution $\bar{\mathbf{x}} = \mathbf{0}$ iff $\text{rank}(\mathbf{A}) = n$.

Corollary for *non-homogeneous* systems:

Let a system $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a nonempty set of solutions where $\mathbf{A} \in \mathbb{R}^{m \times n}$ is a matrix of rank r . Then all solutions of $\mathbf{A}\mathbf{x} = \mathbf{b}$ can be written as $\mathbf{x} = \mathbf{y} + p_1\bar{\mathbf{x}}_1 + p_2\bar{\mathbf{x}}_2 + \cdots + p_{n-r}\bar{\mathbf{x}}_{n-r}$, where

- ▶ p_1, \dots, p_{n-r} are arbitrary real parameters,
- ▶ $\bar{\mathbf{x}}_1, \dots, \bar{\mathbf{x}}_{n-r}$ are suitable solutions of $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$, and
- ▶ \mathbf{y} is an arbitrary solution of the system $\mathbf{A}\mathbf{x} = \mathbf{b}$.

Solutions of systems $\mathbf{Ax} = \mathbf{b}$ — summary and example

1. Transform the augmented matrix $(\mathbf{A}|\mathbf{b})$ into the echelon form:

$$(\mathbf{A}|\mathbf{b}) = \left(\begin{array}{ccccc|c} 1 & 4 & 3 & 2 & 1 & 1 \\ 2 & 8 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 6 & 9 & 5 \\ 2 & 8 & 7 & 6 & 3 & 3 \end{array} \right) \sim \sim \left(\begin{array}{ccccc|c} 1 & 4 & 3 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) = (\mathbf{A}'|\mathbf{b}')$$

2. If a pivot is in the last column, then no solution exists.
3. Otherwise solve first the homogeneous system $\mathbf{A}'\bar{\mathbf{x}} = \mathbf{0}$, i.e.

$$\begin{aligned} \bar{x}_5 &= 0 \\ \bar{x}_3 &= -2\bar{x}_4 - \bar{x}_5 &= -2\bar{x}_4 \\ \bar{x}_1 &= -4\bar{x}_2 - 3\bar{x}_3 - 2\bar{x}_4 - \bar{x}_5 &= -4\bar{x}_2 + 4\bar{x}_4 \end{aligned}$$

4. Replace the free variables in $\bar{\mathbf{x}}$ with parameters:

$$\bar{\mathbf{x}} = p_1(-4, 1, 0, 0, 0)^T + p_2(4, 0, -2, 1, 0)^T$$

5. Finally, find any solution of the non-homogeneous system

$\mathbf{A}'\mathbf{x} = \mathbf{b}'$, e.g. $\mathbf{y} = \left(4, -1, 0, \frac{1}{3}, \frac{1}{3}\right)^T$ and get:

$$\mathbf{x} = \left(4, -1, 0, \frac{1}{3}, \frac{1}{3}\right)^T + p_1(-4, 1, 0, 0, 0)^T + p_2(4, 0, -2, 1, 0)^T$$

Verification

Substitute \mathbf{x} *including parameters* into the *original* system $\mathbf{Ax} = \mathbf{b}$,
ie., verify \mathbf{y} for $\mathbf{Ax} = \mathbf{b}$ and also $\bar{\mathbf{x}}_1, \dots, \bar{\mathbf{x}}_{n-r}$ for $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$.

Example:

$$\begin{aligned} \text{For a system} \quad & x_1 + 4x_2 + 3x_3 + 2x_4 + x_5 = 1 \\ & 2x_1 + 8x_2 + 4x_3 = 0 \\ & 3x_3 + 6x_4 + 9x_5 = 5 \\ & 2x_1 + 8x_2 + 7x_3 + 6x_4 + 3x_5 = 3 \end{aligned}$$

substitute

$$\begin{aligned} \mathbf{x} &= \left(4, -1, 0, \frac{1}{3}, \frac{1}{3}\right)^T + p_1(-4, 1, 0, 0, 0)^T + p_2(4, 0, -2, 1, 0)^T \\ &= \left(4 - 4p_1 + 4p_2, -1 + p_1, -2p_2, \frac{1}{3} + p_2, \frac{1}{3}\right)^T \text{ as follows:} \end{aligned}$$

$$\begin{aligned} (4 - 4p_1 + 4p_2) + 4(p_1 - 1) + 3(-2p_2) + 2\left(\frac{1}{3} + p_2\right) + \frac{1}{3} &= 1 \\ 2(4 - 4p_1 + 4p_2) + 8(p_1 - 1) + 4(-2p_2) &= 0 \\ 3(-2p_2) + 6\left(\frac{1}{3} + p_2\right) + 9 \cdot \frac{1}{3} &= 5 \\ 2(4 - 4p_1 + 4p_2) + 8(p_1 - 1) + 7(-2p_2) + 6\left(\frac{1}{3} + p_2\right) + 3 \cdot \frac{1}{3} &= 3 \end{aligned}$$

Observe that the parameters on the left side will cancel each other.

What the tests verify

If we do not insert the solution \mathbf{x} including the parameters into the original system $\mathbf{Ax} = \mathbf{b}$, we may not detect the following errors:

- ▶ Without the parameters we verify only the solution \mathbf{y} .
- ▶ A specific choice of parameters verifies only one solution, not all. We wouldn't have to reveal faulty solutions.

Example: We check $\mathbf{x} = (1, 3)^T + p_1(1, 1)^T$ for $2x_1 + x_2 = 8$.

For $p_1 = 1$, we have $\mathbf{x} = (2, 4)^T$, which satisfies $2 \cdot 2 + 4 = 8$, but neither $\mathbf{y} = (1, 3)^T$ nor choices of p_1 are valid solutions.

- ▶ Substitution into $\mathbf{A}'\mathbf{x} = \mathbf{b}'$ verifies the correctness of the backward substitution, but not of Gaussian elimination.

Warning: We cannot yet verify the completeness of the solution set. It may happen that we add a new condition due to an error in Gaussian elimination. Then we get only a *subset* of all solutions.

E.g. on error in the last column we may not obtain any solution:

$$\left(\begin{array}{cccc|c} 1 & 4 & 3 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & -2 & -4 & -2 & -2 \\ 0 & 0 & 3 & 6 & 9 & 5 \end{array} \right) \sim_{+2\text{II}} \left(\begin{array}{cccc|c} 1 & 4 & 3 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 6 & 9 & 5 \end{array} \right) \checkmark \text{ vs. } \left(\begin{array}{cccc|c} 1 & 4 & 3 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & -2 & -4 & -2 & -2 \\ 0 & 0 & 3 & 6 & 9 & 5 \end{array} \right) \sim_{+2\text{II}} \left(\begin{array}{cccc|c} 1 & 4 & 3 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 3 & 6 & 9 & 5 \end{array} \right) \mathbf{x}$$

Reduced row echelon form

Definition: A row echelon form of a matrix is *reduced* if each pivot is 1 and all other elements in pivot columns are 0es.

Fact: Every matrix A has a unique A' in reduced row echelon form such that $A \sim \sim A'$. "Pf": distinct r.r.e.f. yields different solutions.

Any matrix in row echelon form can be reduced by:

- ▶ Divide each row by $a_{i,j(i)}$, thus get 1s as pivots.
- ▶ Foreach $i = r, \dots, 1$, eliminate each $a_{i',j(i)}$ with $i' < i$ by addition $-a_{i',j(i)}$ -multiple of the i -th row to i' -th row. thus to get 0es above the pivot $a_{i,j(i)}$

Example:

$$\begin{pmatrix} 1 & 4 & 3 & 2 & 1 & | & 1 \\ 0 & 0 & 1 & 2 & 1 & | & 1 \\ 0 & 0 & 0 & 0 & 6 & | & 2 \end{pmatrix} \underset{:6}{\sim} \begin{pmatrix} 1 & 4 & 3 & 2 & 1 & | & 1 \\ 0 & 0 & 1 & 2 & 1 & | & 1 \\ 0 & 0 & 0 & 0 & 1 & | & \frac{1}{3} \end{pmatrix} \underset{-III}{\sim} \underset{-III}{\sim} \begin{pmatrix} 1 & 4 & 3 & 2 & 0 & | & \frac{2}{3} \\ 0 & 0 & 1 & 2 & 0 & | & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 1 & | & \frac{1}{3} \end{pmatrix} \underset{-3III}{\sim} \begin{pmatrix} 1 & 4 & 0 & -4 & 0 & | & -\frac{4}{3} \\ 0 & 0 & 1 & 2 & 0 & | & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 1 & | & \frac{1}{3} \end{pmatrix}$$

This process is sometimes called the *Gauss-Jordan elimination*.

Advantages of reduced row echelon form

$$\mathbf{x} = \left(-\frac{4}{3}, 0, \frac{2}{3}, 0, \frac{1}{3}\right)^T + p_1(-4, 1, 0, 0, 0)^T + p_2(4, 0, -2, 1, 0)^T$$

$$\left(\begin{array}{ccccc|c} 1 & 4 & 3 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 4 & 0 & -4 & 0 & -\frac{4}{3} \\ 0 & 0 & 1 & 2 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 1 & 4 & 0 & -4 & 0 & -\frac{4}{3} \\ 0 & 0 & 1 & 2 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \end{array}\right)$$

yields directly the solution

$$\mathbf{y} = \left(-\frac{4}{3}, 0, \frac{2}{3}, 0, \frac{1}{3}\right)^T$$

by choosing $y_2 = y_4 = 0$

$$\left(\begin{array}{ccccc|c} 1 & 4 & 0 & -4 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

yields $\bar{\mathbf{x}}_1 = (-4, 1, 0, 0, 0)^T$ solving the system $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$ by the choice $(\bar{\mathbf{x}}_1)_2 = 1$ and $(\bar{\mathbf{x}}_1)_4 = 0$

$$\left(\begin{array}{ccccc|c} 1 & 4 & 0 & -4 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

yields $\bar{\mathbf{x}}_2 = (4, 0, -2, 1, 0)^T$ solving the system $\mathbf{A}\bar{\mathbf{x}} = \mathbf{0}$ by the choice $(\bar{\mathbf{x}}_2)_2 = 0$ and $(\bar{\mathbf{x}}_2)_4 = 1$

Ill-conditioned system

$$667x_1 - 835x_2 = 168$$

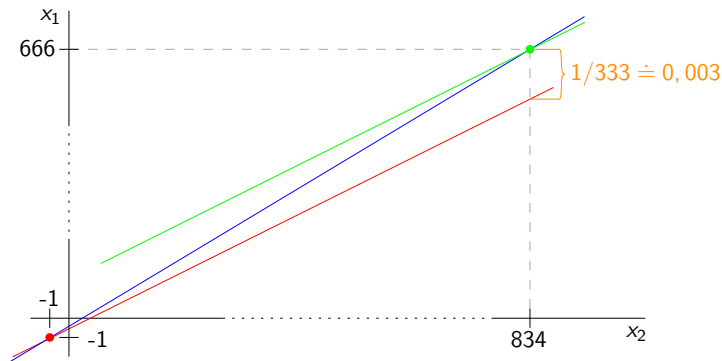
$$266x_1 - 333x_2 = \mathbf{67}$$

solution $\mathbf{x} = (-1, -1)^T$

$$667x_1 - 835x_2 = 168$$

$$266x_1 - 333x_2 = \mathbf{68}$$

solution $\mathbf{x} = (834, 666)^T$



Questions to understand the lecture topic

- ▶ How does composition of mappings relate to the properties "to be injective" and "to be surjective"?
— add details from the proof of the theorem.
- ▶ Why is it possible to express the values of leading variables as *linear* functions of free variables?
- ▶ Is Gauss–Jordan elimination asymptotically computationally as difficult, easier, or more difficult than Gauss elimination with backward substitution?