Topological methods in combinatorics - tutorials

Problem set 4 – The Ham sandwich theorem, homology

Submitted: **22. 04. 2022** - Hints: **06. 05. 2022** - Deadline: **13. 05. 2022**
Submit solution to: skotnica at kam.mff.cuni.cz

1. The *Ham sandwich theorem* says the following:

   Given finite sets $A_1, \dots, A_d$ of points in $\mathbb{R}^d$, there is a hyperplane h that contains at most $\left\lfloor \frac{|A_i|}{2} \right\rfloor$ points from each set A_i in each open half-space determined by h.

   Show that if we replace the original definition of bisecting by more natural definition: a hyperplane h bisects $A \subset \mathbb{R}^d$, if $|h^+ \cap A| + \frac{1}{2}|h \cap A| = \frac{1}{2}|A|$, where h^+ is one of the open half-spaces defined by h, then there exist two finite point sets in the plane which cannot be bisected. **[2]**

2. Prove that any mass distribution in the plane can be dissected into four equal parts by two lines. **[2]**

3. The theorem of Akiyama and Alon says the following:

   Consider sets $A_1, A_2, \dots, A_d$, of n points each in general position in $\mathbb{R}^d$; imagine that the points of A_1 are red, the points of A_2 blue, etc. (each A_i has its own color). Then the points of the union $A_1 \cup \dots \cup A_d$ can be partitioned into "rainbow" d-tuples (each d-tuple contains one point of each color) with disjoint convex hulls.

   Prove the planar case ($d = 2$) of this theorem by considering a perfect red–blue matching with the minimum possible total length of the edges. **[3]**

4. Show, that for each $z \in \mathbb{Z}, d \in \mathbb{N}$ there are triangulations K_1 and K_2 of the sphere S^d and a simplicial map $f\colon K_1 \to K_2$ such that $\deg f = z$. **[3]**

5. Let K be a simplicial complex such that $|K|$ is path connected. Show that $H_0(K; \mathbb{Z}) = \mathbb{Z}$. **[3]**

Information about tutorials `https://kam.mff.cuni.cz/~dbulavka/teaching/ss2122/topomethods.html`