

Topological methods in combinatorics - tutorials

Problem set 3 – The Borsuk-Ulam Theorem and its application

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Submit solution to: skotnica at kam.mff.cuni.cz

1. Prove directly the 1-dimensional version of Lyusternik-Shnirelman theorem (LS-o). In other words, prove, that for each covering of S^1 by two open sets there is a pair of antipodal points in one of these two sets. [2]
2. Let the torus be represented as $T = S^1 \times S^1 \in \mathbb{R}^4$.
 - (a) Show that there is a continuous map $f: T \rightarrow \mathbb{R}^2$ for which there is no $x \in T$ such that $f(x) = f(-x)$. [2]
 - (b) Show that for each continuous $f: T \rightarrow \mathbb{R}$ there is x such that $f(x) = f(-x)$. [1]
3. Show that the following statement is equivalent to one of the versions of the Borsuk-Ulam theorem: Whenever S^n is covered by $n+1$ sets A_1, A_2, \dots, A_{n+1} , each A_i open or closed, there is an i such that $A_i \cap (-A_i) \neq \emptyset$. [3]
4. Show that the following statement is equivalent to one of the versions of the Borsuk-Ulam theorem: If $f: S^n \rightarrow S^n$ is antipodal, then every mapping $g: S^n \rightarrow S^n$ which is homotopic to f is surjective. [2+2]
5. Let $SG(n, k)$ denote *Schrijver graph* whose vertices are stable k -element subsets of $\{1, \dots, n\}$ ¹ and whose two vertices form an edge if and only if the corresponding k -element subsets are disjoint.
 - (a) Show that the Schrijver graph $SG(n, k)$ is not regular in general; that is, its vertices need not all have the same degree. [2]
 - (b) Show that not all $SG(n, k)$ are edge-critical (an edge may be removed without decreasing the chromatic number). [3]

¹Recall that $S \subseteq \{1, \dots, n\}$ is *stable* if it does not contain any two adjacent elements modulo n . That is, if $i \in S$, then $i+1 \notin S$, and if $n \in S$, then $1 \notin S$.