

Topological methods in combinatorics - tutorials

Problem set 2 – Homotopy and simplicial complexes

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1. (Elementary collapse) Let K be a simplicial complex and let $\sigma \in K$ contained in a unique maximal face $\tau \in K$, $\sigma \subsetneq \tau$. By an *elementary collapse* we mean an operation that produces a new complex K' from K . The faces of K' are the faces of K except by $\{\mu \in K : \sigma \subseteq \mu \subseteq \tau\}$. Show that K and K' are homotopically equivalent. [3]
2. (Barycentric subdivision) Let K be a simplicial complex, show that the geometric realization of K and its barycentric subdivision are homeomorphic, i.e. $|K|$ and $|\text{sd}(K)|$ are homeomorphic. [4]
3. Let G be a bipartite graph with parts A and B . Let N_A be the simplicial complex with vertex set A and a subset of A is a face of N_A if these vertices have a common neighbor in B . We define N_B analogously. Prove that the spaces $|N_A|$ and $|N_B|$ are homotopy equivalent. [3]
4. (Diameter of barycentric subdivision)
 - (a) Prove that the diameter of an arbitrary simplex σ is equal to the distance between some two vertices of σ . [2]
 - (b) Prove that for every n and $\delta > 0$ there exists k such that if σ^n is any n -dimensional simplex of diameter 1, then all the simplices of $\text{sd}^k(\sigma^n)$, i.e. barycentric subdivision iterated k times, have diameter at most δ . Does k have to depend on n ? [2]
5. Let $s: K \rightarrow L$ and $l: L \rightarrow Z$ be a simplicial maps.
 - (a) Show, in detail, that the affine extension $|s|: |K| \rightarrow |L|$ is a continuous function. [2]
 - (b) Show that $|l \circ s| = |l| \circ |s|$. [1]
6. Let K and L be simplicial complexes, show that $|K * L|$ and $|K| * |L|$ are homeomorphic topological spaces. [3]