Topological methods in combinatorics - tutorials

Problem set 1 – Basics of general and algebraic topology

Submitted: **25.02.2022** - Hints: **11. 03. 2021** - Deadline: **18. 03. 2021**
Submit solution to: dbulavka+tm at kam.mff.cuni.cz

Please include with your solutions a statement if you want your score to be displayed on the website as well as a nickname.

1. Let X and Y be topological spaces, $f\colon X \to Y$ continuous function and $M, N \subset X$. Decide if the following claims hold, justify in each case.
   (a) If M is closed set, then $f(M)$ is closed. **[1]**
   (b) If M is open set, then $f(M)$ is open. **[1]**
   (c) If M is connected then $f(M)$ is connected. **[1]**
   (d) If M is disconnected, then $f(M)$ is disconnected. **[1]**
   (e) If M is closed and N compact, then $M \cap N$ is compact. **[1]**
2. Let $v_1, \ldots, v_n$ be the vertices of a convex polygon P in $\mathbb{R}^2$ with the origin in the interior. Show that ∂P and S^1 are homeomoprhic. **[3]**
3. Show that the topology on $\mathbb{R}^n$ given by the basis
   $$\{B(x, \epsilon)\colon x \in \mathbb{R}^n, \epsilon \in \mathbb{R} \text{ such that } \epsilon > 0\}$$
   coincides with the one given by $\epsilon - \delta$ definition. Here $B(x, \epsilon)$ is the open ball with center x and radius ϵ. **[2]**
4. Let $1 \leq r \leq n/2$, the Kneser graph $KG_{n,r}$ is the graph with vertex set all the r-subsets of $[n]$, i.e. $V(KG_{n,r}) = \{S \subset [n]\colon |S| = r\}$. A pair of r-sets S, T form an edge in $KG_{n,r}$ if $S \cap T = \emptyset$. Show that the chromatic number of $KG_{n,r}$ is at most $n - 2r + 2$. **[2]**
5. (Gluing lemma for continuous functions) Let X be a topological space and $A_1, \ldots, A_n$ closed subspaces of X such that $X = \cup_{i=1}^n A_i$. Let $f\colon X \to Y$ function between topological spaces. Show that f is continuous if and only if the restriction of f to each A_i is continuous. **[2]**
6. Show that the connected components of a topological space X are closed subsets. **[2]**
7. Show that two connected graphs with the same number of vertices and edges are homotopically equivalent. **[4]**

Information about tutorials `https://kam.mff.cuni.cz/~dbulavka/teaching/ss2122/topomethods.html`