# Topological methods in combinatorics - tutorials

Class work 3 – The Borsuk-Ulam Theorem and its application

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**Definition 1.** Let $\mathcal{F} \subseteq 2^X$ be a system of subsets of $X$.

The chromatic number $\chi(\mathcal{F})$ of $\mathcal{F}$ is the minimum $m$ such that there exists a proper coloring $c\colon X \to \{1, \ldots, m\}$. That is, $|c(F)| > 1$ for each $F \in \mathcal{F}$.

Let the *m-colorability defect*, denoted by $cd_m(\mathcal{F})$, be the minimum size of a subset $Y \subseteq X$ such that the system of the sets of $\mathcal{F}$ that contain no points of $Y$ can be properly colored by $m$ colors.

Let $\mathrm{KG}(\mathcal{F})$ be a graph whose vertices are sets from $\mathcal{F}$ and whose vertices form an edge if and only the corresponding sets are disjoint.

**Theorem 2** (Dolnikov's theorem)**.** *For any finite set system $\mathcal{F} \subseteq X$*

$$\chi(\mathrm{KG}(\mathcal{F})) \geq \mathrm{cd}_2(\mathcal{F}).$$

**Definition 3.** Let $\mathrm{SG}(n, k)$ denote *Schrijver graph* whose vertices are stable $k$-element subsets of $\{1, \ldots, n\}$. (Recall that $S \subseteq \{1, \ldots, n\}$ is *stable* if it does not contain any two adjacent elements modulo $n$. That is, if $i \in S$, then $i + 1 \notin S$, and if $n \in S$, then $1 \notin S$.)

1. Prove that if $f\colon X \to Y$ is a continuous mapping to a contractible space $Y$, then it is nullhomotopic.
2. Let $f\colon S^n \to S^n$ be a continuous mapping which is not surjective. Prove that $f$ is nullhomotopic.
3. Let $f\colon S^n \to Y$ be a continuous mapping. Prove that the following statements are equivalent.
   (a) $f$ is nullhomotopic.
   (b) $f$ can be continuously extended to $B^{n+1}$.
4. For set systems $\mathcal{F}$ with $\chi(\mathrm{KG}(\mathcal{F})) \leq 2$ prove Dolnikov's theorem by direct combinatorial argument.
5. Show that every graph is $\mathrm{KG}(\mathcal{F})$ for some set system $\mathcal{F}$.
6. What is the number of vertices of $\mathrm{SG}(n, k)$?
7. Show that $\mathrm{SG}(n, k)$ is vertex-critical for chromatic number. In other words, if we remove an arbitrary vertex from $\mathrm{SG}(n, k)$, the chromatic number decreases.

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Information about tutorials `https://kam.mff.cuni.cz/~dbulavka/teaching/ss2122/topomethods.html`