

Topological methods in combinatorics - tutorials

Class work 2 – Homotopy and simplicial complexes

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Definition 1. Let X, Y be two topological spaces, the join of X and Y is the quotient space $X \times Y \times I / \simeq$ where the relation is given by $(x, y, 0) \simeq (x', y, 0)$ and $(x, y, 1) \simeq (x, y', 1)$ for all $x, x' \in X$ and $y, y' \in Y$.

For abstract simplicial complexes K and L , the join is defined by $K * L = \{\sigma * \tau : \sigma \in K, \tau \in L\}$ where we consider the vertex sets being disjoint.

For a (finite) family of sets \mathcal{F} , the nerve of \mathcal{F} , $\mathcal{N}(\mathcal{F})$ is the simplicial complex with vertices being the elements of \mathcal{F} and $\sigma \subset \mathcal{F}$ is a face if $\bigcap_{F \in \sigma} F \neq \emptyset$.

Lemma 2 (Nerve lemma). *Let K_1, K_2, \dots, K_n be subcomplexes of a finite simplicial complex K that together cover K (each simplex of K is in at least one K_i). Suppose that the intersection $\bigcap_{i \in J} |K_i|$ is empty or contractible for each nonempty $J \subset [n]$. Then the geometric realization of the nerve of $\{|K_1|, \dots, |K_n|\}$ is homotopy equivalent to $|K|$.*

Reminder. The distance function $|x - y|$ is convex, i.e. $|tx_1 + (1 - t)x_2 - y| \leq t|x_1 - y| + (1 - t)|x_2 - y|$.

1. Describe the nerve of the maximal faces of a square, a triangle and a tetrahedron.
2. What is the relation between the nerve of all the faces of a simplicial complex and the nerve of only its maximal faces?
3. Show that $S^1 \times S^1$ and $I \times I / \{(x, 0) \simeq (x, 1), (0, y) \simeq (1, y)\}$ are homeomorphic.
4. Show that the Möbius strip $M = I \times I / \{(x, 0) \simeq (1 - x, 1)\}$ is homotopical equivalent to S^1 .
5. Let σ be 2-simplex and $\text{sd}(\sigma)$ its barycentric subdivision. Show that $|\sigma|$ and $|\text{sd}(\sigma)|$ are homeomorphic.
6. Let σ be geometric 2-simplex, show that its diameter is at most the distance of some 2 vertices.
7. Prove that a topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \in X \times X : x \in X\}$ is closed in $X \times X$. [3]
8. Let \simeq be an equivalence relation on a topological space X such that the quotient map $q: X \rightarrow X / \simeq$ is open. Prove that X / \simeq is Hausdorff if and only if $\simeq \subset X \times X$ is a closed subset