

Algorithmic game theory

Martin Balko

3rd lecture

October 19th 2026



Proof of the Minimax Theorem

The Minimax Theorem

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The Minimax Theorem (Theorem 2.21)

Every zero-sum game has worst-case optimal strategies, computable in polynomial time. There is a number v such that, for any worst-case optimal strategies x^* and y^* , the strategy profile (x^*, y^*) is a Nash equilibrium and $\beta(x^*) = (x^*)^\top M y^* = \alpha(y^*) = v$.



Figure: John von Neumann (1903–1957) and Oskar Morgenstern (1902–1977).

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- Recall that $\beta(x) = \min_{y \in S_2} x^\top M y$ and $\alpha(y) = \max_{x \in S_1} x^\top M y$ are the best possible cost of player 2 to x and payoff of player 1 to y , respectively.

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- We prove the theorem using **linear programming**.

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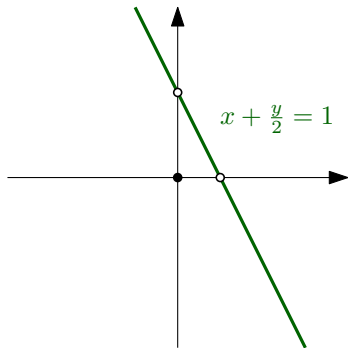
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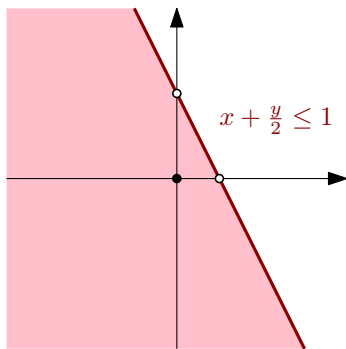
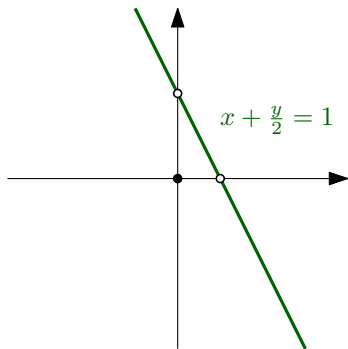
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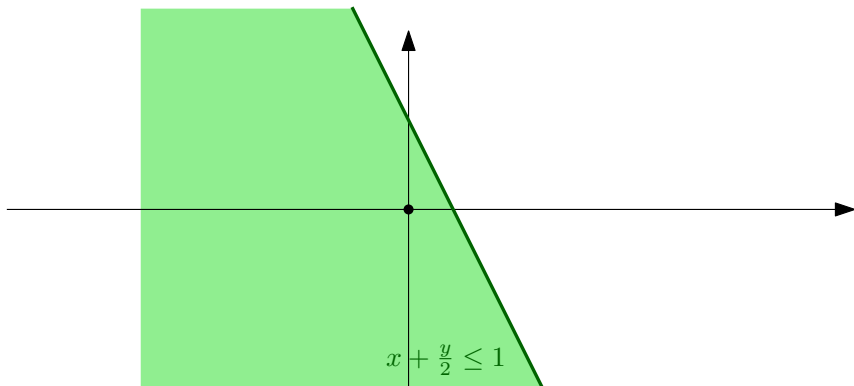
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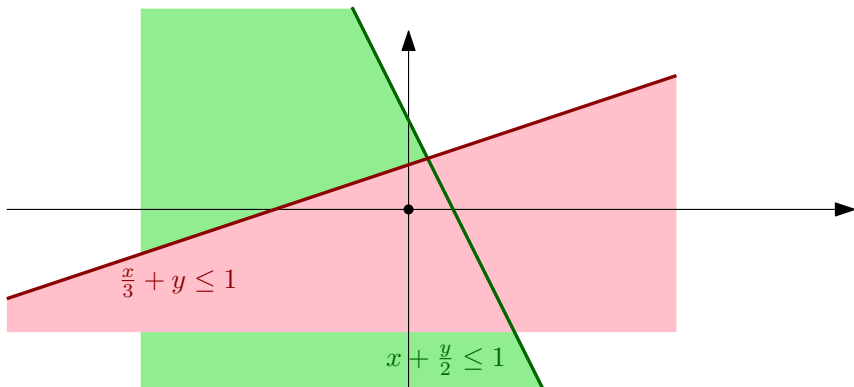
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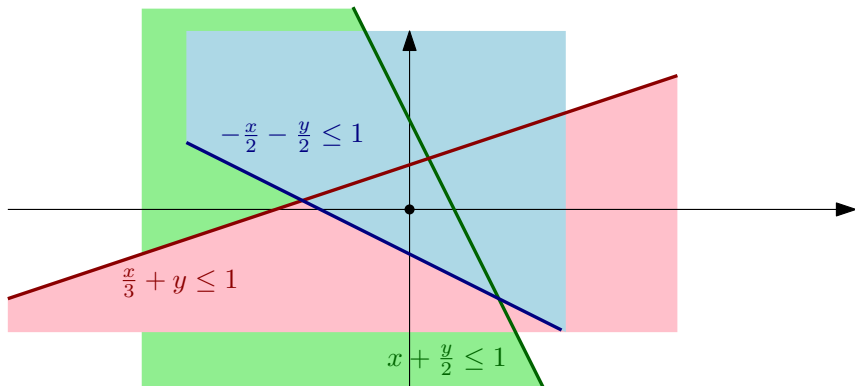
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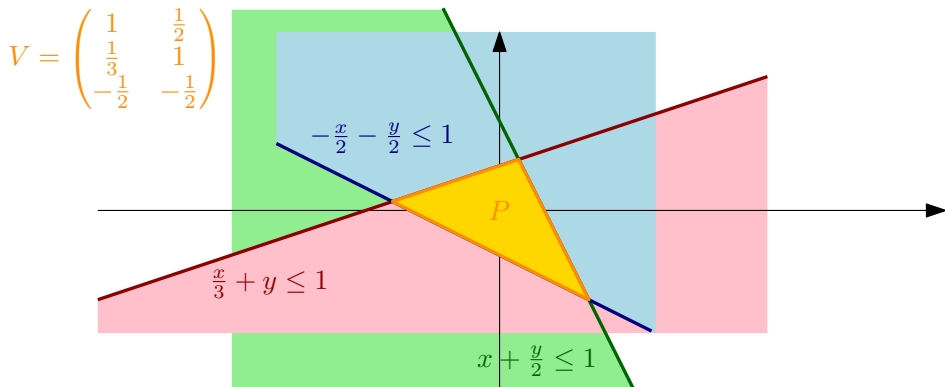
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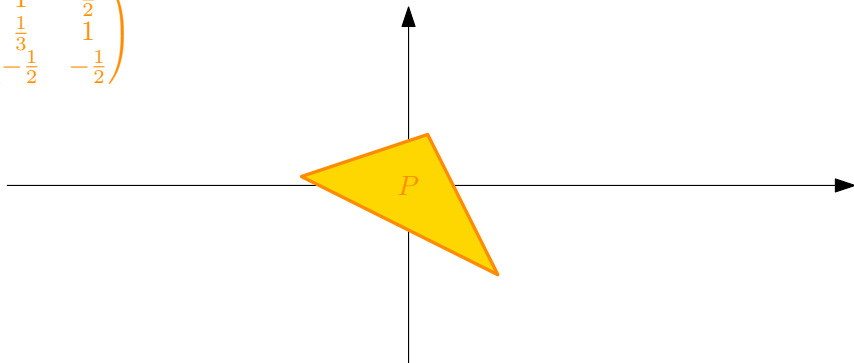
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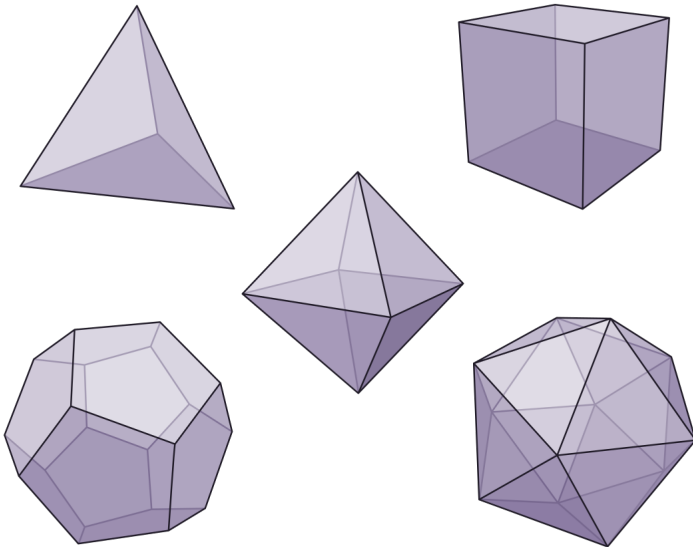
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$$V = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{3} & 1 \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$



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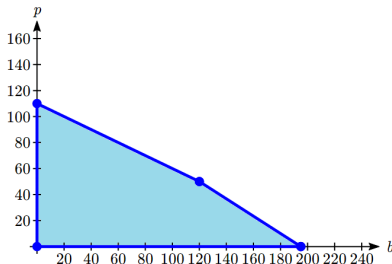
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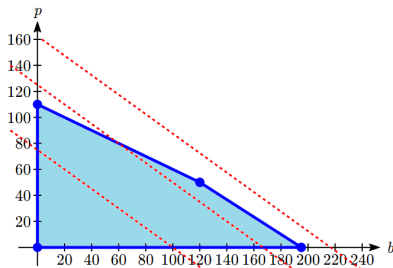
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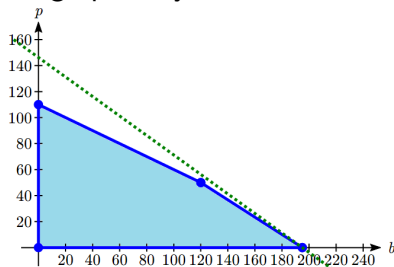
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- Dual programs can be constructed for any linear program.

General recipe for duality

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	Primal linear program	Dual linear program
Variables	$x_1, \dots, x_m$	$y_1, \dots, y_n$
Matrix	$A \in \mathbb{R}^{n \times m}$	$A^T \in \mathbb{R}^{m \times n}$
Right-hand side	$b \in \mathbb{R}^n$	$c \in \mathbb{R}^m$
Objective function	$\max c^T x$	$\min b^T y$
Constraints	i th constraint has $\leq$ $\geq$ $=$ $x_j \geq 0$ $x_j \leq 0$ $x_j \in \mathbb{R}$	$y_i \geq 0$ $y_i \leq 0$ $y_i \in \mathbb{R}$ j th constraint has $\geq$ $\leq$ $=$

Table: A recipe for making dual programs.

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- Naive straightforward approach with variables $x_1, \dots, x_m$:

$$\text{maximize } \beta(x) \text{ subject to the constraints } \sum_{i=1}^m x_i = 1 \text{ and } x \geq 0.$$

- **In general, this is not LP!** (the objective function $\beta(x) = \min_{y \in S_2} x^\top M y$ is not linear in x) What can we compute with LP?

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Nash equilibria in bimatrix games

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	Testify	Remain silent
Testify	$(-2, -2)$	$(0, -3)$
Remain silent	$(-3, 0)$	$(-1, -1)$

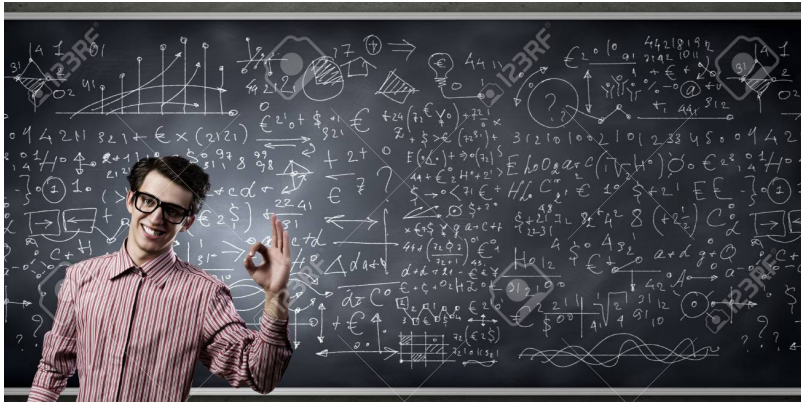


Bimatrix games examples: collaborative projects



Source: <https://filestage.io/>

Bimatrix games examples: education, knowledge sharing



Source: <https://www.123rf.com/>

Bimatrix games examples: the battle for Gotham's soul

	Cooperate	Detonate
Cooperate	(0,0)	(0,1)
Detonate	(1,0)	(0,0)



Sources: <https://www.cbr.com/>

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Source: <https://pinterest.com>

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- Later, we show a **classical algorithm** for this problem.

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In a normal-form game $G = (P, A, u)$ of n players, for every player $i \in P$, a mixed strategy s_i is a best response to s_{-i} if and only if all pure strategies in the support of s_i are best responses to s_{-i} .

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- In bimatrix games, we can **enumerate supports** and test each pair using LP.
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- Once we have the right supports, the precise mixed strategies can be computed by solving a system of algebraic equations and inequalities (linear for two players).

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- The running time is then about 4^n for $m = n$.

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$$-y_2 = u, \quad y_1 - 10y_2 = u, \quad y_1 + y_2 = 1$$

- This yields a unique solution $(x_1, x_2) = (\frac{9}{10}, \frac{1}{10})$ and $(y_1, y_2) = (\frac{9}{10}, \frac{1}{10})$.

Example: Game of chicken

- We show the Support enumeration on the **Game of chicken**.

	Turn (1)	Go straight (2)
Turn (1)	(0, 0)	(-1, 1)
Go straight (2)	(1, -1)	(-10, -10)

- That is, we have $M = \begin{pmatrix} 0 & -1 \\ 1 & -10 \end{pmatrix}$ and $N = \begin{pmatrix} 0 & 1 \\ -1 & -10 \end{pmatrix} = M^T$.
- If $I = \{1, 2\}$ and $J = \{1, 2\}$, then we want to solve the following system of 6 equations with 6 variables x_1, x_2, y_1, y_2, u, v :

$$-x_2 = v, \quad x_1 - 10x_2 = v, \quad x_1 + x_2 = 1$$

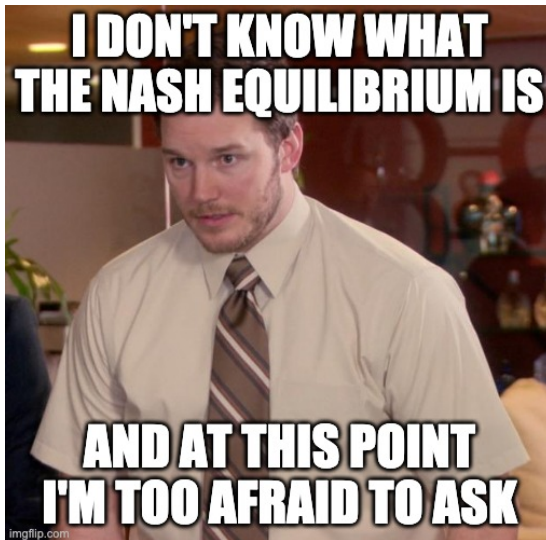
$$-y_2 = u, \quad y_1 - 10y_2 = u, \quad y_1 + y_2 = 1$$

- This yields a unique solution $(x_1, x_2) = (\frac{9}{10}, \frac{1}{10})$ and $(y_1, y_2) = (\frac{9}{10}, \frac{1}{10})$. Since $x, y > 0$ and there is no better pure strategy, we have NE.

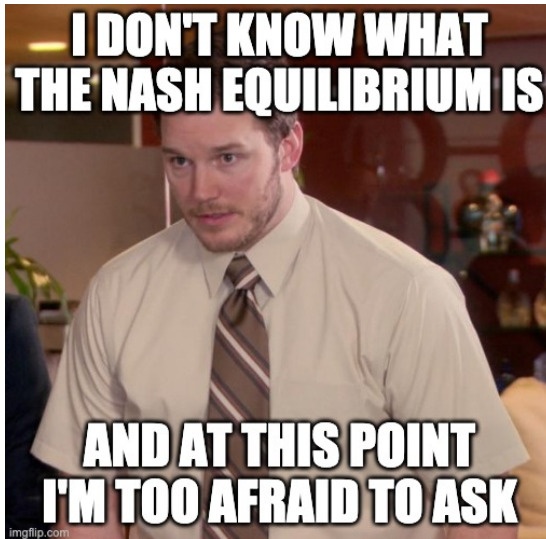


- Next lecture we learn the **Lemke–Howson algorithm**, a classical algorithm to find a Nash equilibrium in bimatrix games.

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Thank you for your attention.