

Algorithmic game theory – Tutorial 4*

1 Bimatrix games

A *bimatrix game* is a normal-form game of 2 players. A bimatrix game is *non-degenerate* if every player has at most k pure best responses to every strategy with support of size k . A *zero-sum bimatrix game* is a game where the utility of one player equals the loss of the other one. For a bimatrix game $G = (\{1, 2\}, A, u)$ with $A_1 = \{1, \dots, m\}$ and $A_2 = \{1, \dots, n\}$, we use the payoff matrices M and N where $(M)_{i,j} = u_1(i, j)$ and $(N)_{i,j} = u_2(i, j)$ for all $i \in A_1$ and $j \in A_2$.

The following algorithm for computing NE in non-degenerate games was shown at the lecture.

Algorithm 1.1: SUPPORT ENUMERATION(G)

Input: A non-degenerate game G .

Output: All Nash equilibria of G .

for every $k \in \{1, \dots, \min\{m, n\}\}$ and a pair of supports (I, J) of size k
 $\left\{ \begin{array}{l} \text{Solve the system of equations } \sum_{i \in I} (N^\top)_{j,i} x_i = v, \sum_{j \in J} (M)_{i,j} y_j = u, \\ \text{for all } i \in I, j \in J \text{ and } \sum_{i \in I} x_i = 1, \sum_{j \in J} y_j = 1. \\ \text{if } x_i > 0 \text{ (} i \in I \text{), } y_j > 0 \text{ (} j \in J \text{) and } u = \max\{(M)_{i,y} : i \in A_1\}, \\ v = \max\{(N^\top)_{j,x} : j \in A_2\}, \text{ then return } (x, y) \text{ as a Nash equilibrium} \end{array} \right.$

Exercise 1. Use the Support enumeration algorithm to find a Nash equilibrium of the Battle of Sexes with supports of size 2.

	Football (1)	Opera (2)
Football (1)	(2, 1)	(0, 0)
Opera (2)	(0, 0)	(1, 2)

Table 1: The Battle of Sexes.

Exercise 2. Prove that the Game for Gotham's soul is degenerate. Use the equations from the Support enumeration algorithm to find a Nash equilibrium with supports $I = \{2\}$ and $J = \{1, 2\}$.

	Cooperate (1)	Detonate (2)
Cooperate (1)	(0, 0)	(0, 1)
Detonate (2)	(1, 0)	(0, 0)

Table 2: The Battle for Gotham's soul.

Exercise 3. Decide which of these two bimatrix games is degenerate.

(a) $M = \begin{pmatrix} 0 & 4 & 1 \\ 2 & 2 & 4 \\ 3 & 2 & 2 \end{pmatrix}$ and $N = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 1 & 1 \end{pmatrix}$.

(b) $M = \begin{pmatrix} 0 & 4 & 4 \\ 2 & 2 & 1 \\ 3 & 2 & 2 \end{pmatrix}$ and $N = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{pmatrix}$.

*Information about the course can be found at <http://kam.mff.cuni.cz/~balko>

Exercise 4. Prove that the following linear programs from the proof of the Minimax theorem are dual to each other.

(a) For a matrix $M \in \mathbb{R}^{m \times n}$ and a fixed vector $x \in \mathbb{R}^m$,

	Program P	Program D
Variables	$y_1, \dots, y_n$	$x_0 \in \mathbb{R}$
Objective function	$\min x^\top M y$	$\max x_0$
Constraints	$\sum_{j=1}^n y_j = 1,$ $y_1, \dots, y_n \geq 0.$	$\mathbf{1}x_0 \leq M^\top x.$

(b) For a matrix $M \in \mathbb{R}^{m \times n}$,

	Program P'	Program D'
Variables	$y_0 \in \mathbb{R}, y_1, \dots, y_n$	$x_0 \in \mathbb{R}, x_1, \dots, x_m$
Objective function	$\min y_0$	$\max x_0$
Constraints	$\mathbf{1}y_0 - M y \geq \mathbf{0},$ $\sum_{j=1}^n y_j = 1,$ $y_1, \dots, y_n \geq 0.$	$\mathbf{1}x_0 - M^\top x \leq \mathbf{0},$ $\sum_{i=1}^m x_i = 1,$ $x_1, \dots, x_m \geq 0.$

You can use the following recipe for duality.

	Primal	Dual
Variables	$\mathbf{x} = (x_1, \dots, x_m)$	$\mathbf{y} = (y_1, \dots, y_n)$
Constraint matrix	$A \in \mathbb{R}^{n \times m}$	$A^\top \in \mathbb{R}^{m \times n}$
Right-hand side	$\mathbf{b} \in \mathbb{R}^n$	$\mathbf{c} \in \mathbb{R}^m$
Objective function	$\max \mathbf{c}^\top \mathbf{x}$	$\min \mathbf{b}^\top \mathbf{y}$
Constraints	i th constraint has $\leq$	$y_i \geq 0$
	$\geq$	$y_i \leq 0$
	$=$	$y_i \in \mathbb{R}$
	$x_j \geq 0$	j th constraint has $\geq$
	$x_j \leq 0$	$\leq$
	$x_j \in \mathbb{R}$	$=$

Exercise 5 (Weekly challenge). Prove that if a bimatrix game is non-degenerate, then the system of equations in the Support enumeration algorithm has at most one solution with $x_i > 0$ ($i \in I$), $y_j > 0$ ($j \in J$), $u = \max\{(M)_i y : i \in A_1\}$, and $v = \max\{(N^\top)_j x : j \in A_2\}$.

Hint: Prove that if there are more solutions, then we can reduce the support.