Algorithmic game theory

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7th lecture

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Applications of regret minimization

Our notation

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- The cumulative loss of A is $L_A^T = \sum_{t=1}^T \ell_A^t$.
- Given a comparison class $\mathcal{A}_X$ of agents A_i that select a single action i in all steps, we let $L_{min}^T = \min_{i \in X}\{L_{A_i}^T\}$ be the minimum cumulative loss of an agent from $\mathcal{A}_X$.

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- Our goal is to minimize the external regret $R_A^T = L_A^T - L_{min}^T$.

Example

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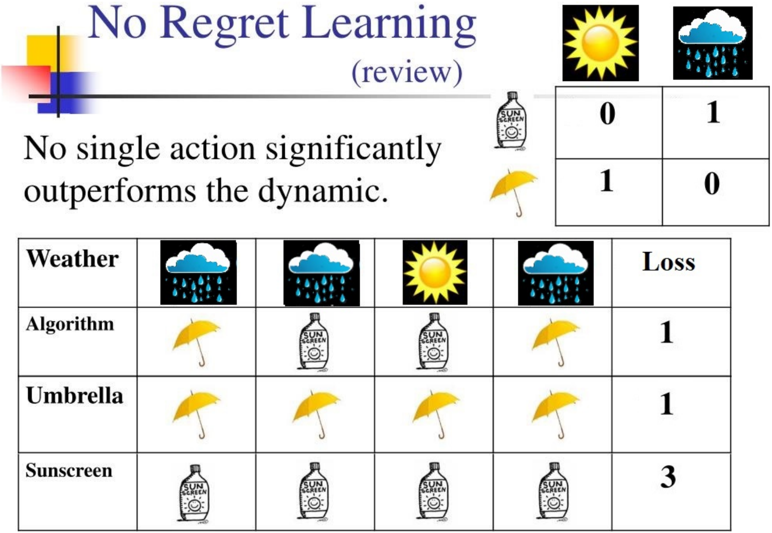
No Regret Learning
(review)
No single action significantly outperforms the dynamic.
0
1
1
0
Weather
Loss
Algorithm
1
Umbrella
1
Sunscreen
3

The Polynomial weights algorithm (PW algorithm)

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Algorithm 0.2: POLYNOMIAL WEIGHTS ALGORITHM(X, T, η)

Input : A set of actions $X = \{1, \ldots, N\}$, $T \in \mathbb{N}$, and $\eta \in (0, 1/2]$.
Output : A probability distribution p^t for every $t \in \{1, \ldots, T\}$.
$w_i^1 \leftarrow 1$ for every $i \in X$,
$p^1 \leftarrow (1/N, \ldots, 1/N)$,
for $t = 2, \ldots, T$

$$\textbf{do} \begin{cases} w_i^t \leftarrow w_i^{t-1}(1 - \eta \ell_i^{t-1}), \\ W^t \leftarrow \sum_{i \in X} w_i^t, \\ p_i^t \leftarrow w_i^t / W^t \text{ for every } i \in X. \end{cases}$$

Output $\{p^t : t \in \{1, \ldots, T\}\}$.

The Polynomial weights algorithm (PW algorithm)

Algorithm 0.3: POLYNOMIAL WEIGHTS ALGORITHM(X, T, η)

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Output $\{p^t \colon t \in \{1, \ldots, T\}\}$.

- For any sequence of loss vectors, we have $R_{\mathrm{PW}}^T \leq 2\sqrt{T \ln N}$.

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Algorithm 0.4: POLYNOMIAL WEIGHTS ALGORITHM(X, T, η)

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Output $\{p^t : t \in \{1, \dots, T\}\}$.

- For any sequence of loss vectors, we have $R^T_{\mathrm{PW}} \leq 2\sqrt{T \ln N}$.
- So the average regret $\frac{1}{T} \cdot R^T_{\mathrm{PW}}$ goes to 0 with $T \to \infty$.

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  - Each player $i \in P$ chooses a mixed strategy $p_i^t = (p_i^t(a_i))_{a_i \in A_i}$ using some algorithm with small external regret such that actions correspond to pure strategies.

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  - Then, i receives a loss vector $\ell_i^t = (\ell_i^t(a_i))_{a_i \in A_i}$, where

$$\ell_i^t(a_i) = \mathbb{E}_{a_{-i}^t \sim p_{-i}^t}[C_i(a_i; a_{-i}^t)]$$

for the product distribution $p_{-i}^t = \prod_{j \neq i} p_j^t$.

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  - That is, $\ell_i^t(a_i)$ is the expected cost of the pure strategy a_i given the mixed strategies chosen by the other players.

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Algorithm 0.7: NO-REGRET DYNAMICS(G, T, ε)

Input : A normal-form game $G = (P, A, C)$ of n players, $T \in \mathbb{N}$, and $\varepsilon > 0$.
Output : A prob. distribution p_i^t on A_i for each $i \in P$ and $t \in \{1, \ldots, T\}$.

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Algorithm 0.8: NO-REGRET DYNAMICS(G, T, ε)

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Output : A prob. distribution p_i^t on A_i for each $i \in P$ and $t \in \{1, \dots, T\}$.
for every step $t = 1, \dots, T$

do $\Bigg\{$

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- "Players play against each other by selecting actions according to an algorithm with small external regret."

Algorithm 0.9: NO-REGRET DYNAMICS(G, T, ε)

Input : A normal-form game $G = (P, A, C)$ of n players, $T \in \mathbb{N}$, and $\varepsilon > 0$.
Output : A prob. distribution p_i^t on A_i for each $i \in P$ and $t \in \{1, \ldots, T\}$.
for every step $t = 1, \ldots, T$
do { Each player $i \in P$ independently chooses a mixed strategy p_i^t using an algorithm with average regret at most ε, with actions corresponding to pure strategies.

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Algorithm 0.10: NO-REGRET DYNAMICS(G, T, ε)

Input : A normal-form game $G = (P, A, C)$ of n players, $T \in \mathbb{N}$, and $\varepsilon > 0$.
Output : A prob. distribution p_i^t on A_i for each $i \in P$ and $t \in \{1, \ldots, T\}$.
for every step $t = 1, \ldots, T$
do
- Each player $i \in P$ independently chooses a mixed strategy p_i^t using an algorithm with average regret at most ε, with actions corresponding to pure strategies.
- Each player $i \in P$ receives a loss vector $\ell_i^t = (\ell_i^t(a_i))_{a_i \in A_i}$, where $\ell_i^t(a_i) \leftarrow \mathbb{E}_{a_{-i}^t \sim p_{-i}^t}[C_i(a_i; a_{-i}^t)]$ for the product distribution $p_{-i}^t = \prod_{j \neq i} p_j^t$.

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Algorithm 0.11: NO-REGRET DYNAMICS(G, T, ε)

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do $\begin{cases} \text{Each player } i \in P \text{ independently chooses a mixed strategy } p_i^t \\ \text{using an algorithm with average regret at most } \varepsilon \text{, with actions} \\ \text{corresponding to pure strategies.} \\ \text{Each player } i \in P \text{ receives a loss vector } \ell_i^t = (\ell_i^t(a_i))_{a_i \in A_i} \text{, where} \\ \ell_i^t(a_i) \leftarrow \mathbb{E}_{a_{-i}^t \sim p_{-i}^t}[C_i(a_i; a_{-i}^t)] \text{ for the product distribution} \\ p_{-i}^t = \prod_{j \neq i} p_j^t. \end{cases}$

Output $\{p^t : t \in \{1, \ldots, T\}\}$.

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- A zero-sum game $G = (\{1, 2\}, A, C)$ with $A_1 = \{a_1, \dots, a_m\}$, $A_2 = \{b_1, \dots, b_n\}$ is represented with an $m \times n$ matrix M where $M_{i,j} = -C_1(a_i, b_j) = C_2(a_i, b_j) \in [-1, 1]$.

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- The expected cost $C_2(s)$ for player 2 equals $x^\top My$, where x and y are the mixed strategy vectors.
- The Minimax theorem then states

$$\max_{x \in S_1} \min_{y \in S_2} x^\top My = \min_{y \in S_2} \max_{x \in S_1} x^\top My.$$

Source: https://www.privatdozent.co/

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- We can prove it without LP!

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- Let $p^1, \dots, p^T$ and $q^1, \dots, q^T$ be strategies played by players 1 and 2.
- We let $\overline{x} = \frac{1}{T}\sum_{t=1}^{T} p^t$ and $\overline{y} = \frac{1}{T}\sum_{t=1}^{T} q^t$ be the time-averaged strategies of players 1 and 2.

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- The payoff vector revealed to each no-regret algorithm after step t is the expected payoff of each strategy, given the mixed strategy played by the other player.

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- With the PW algorithm, we can set $T = 4\ln(\max\{m, n\})/\varepsilon^2$.
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- We let $\overline{x} = \frac{1}{T}\sum_{t=1}^T p^t$ and $\overline{y} = \frac{1}{T}\sum_{t=1}^T q^t$ be the time-averaged strategies of players 1 and 2.
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Modern proof of the Minimax Theorem I

- First, the inequality $\max_x \min_y x^\top My \leq \min_y \max_x x^\top My$ follows easily, since it is only worse to go first.
- Second, we prove the inequality $\max_x \min_y x^\top My \geq \min_y \max_x x^\top My$.
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- The time-averaged expected payoff of 1 is then $v = \frac{1}{T}\sum_{t=1}^{T}(p^t)^\top Mq^t$.

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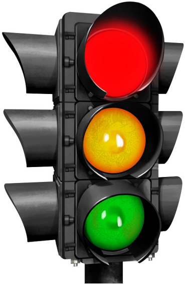
- We define an even more tractable concept and use no-regret dynamics to converge to it.



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- The difference between CCE and CE is that CCE only requires that following your suggested action a_i when a is drawn from p is only a best response in expectation before you see a_i. This makes sense if you have to commit to following your suggested action or not upfront, and do not have the opportunity to deviate after seeing it.

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- Giving probability $1/6$ to each red outcome gives coarse correlated equilibrium in the Rock-Paper-Scissors game.

	Rock	Paper	Scissors
Rock	(0,0)	(-1,1)	(1,-1)
Paper	(1,-1)	(0,0)	(-1,1)
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- Then, the expected payoff of each player is 0 and deviating to any pure strategy gives the expected payoff 0.
- It is not a correlated equilibrium though.

Hierarchy of Nash equilibria

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Pure Nash equilibria,
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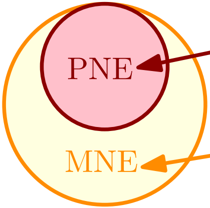

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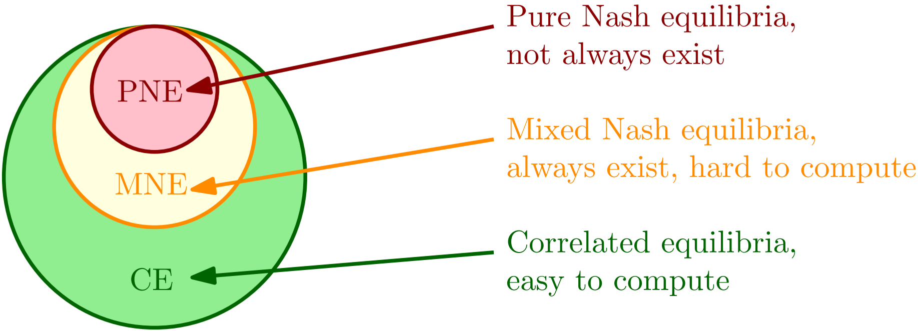
Hierarchy of Nash equilibria



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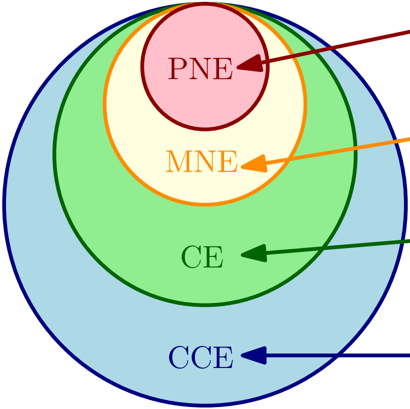

Pure Nash equilibria,
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Correlated equilibria,
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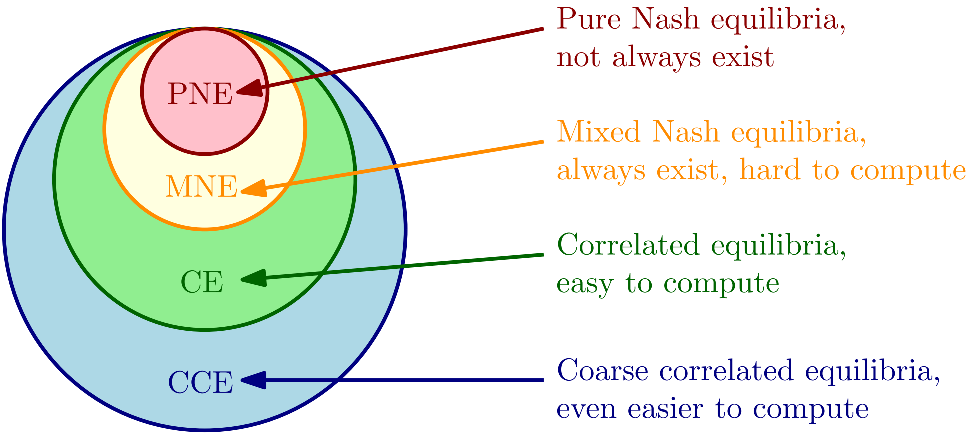
Hierarchy of Nash equilibria



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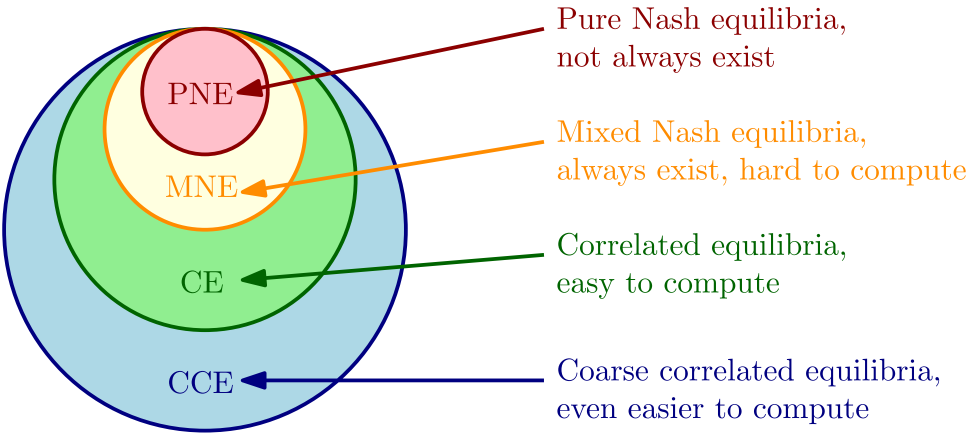

- In general normal-form game, no-regret dynamics converges to a coarse correlated equilibrium.
- For $\varepsilon > 0$, a probability distribution p on A is an ε-coarse correlated equilibrium (ε-CCE) if $\mathbb{E}_{a \sim p}[C_i(a)] \leq \mathbb{E}_{a \sim p}[C_i(a_i'; a_{-i})] + \varepsilon$.

Converging to CCE

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For every $G = (P, A, C)$, $\varepsilon > 0$, and $T = T(\varepsilon) \in \mathbb{N}$, if after T steps of the No-regret dynamics, each player $i \in P$ has time-averaged expected regret at most ε, then p is ε-CCE where $p^t = \prod_{i=1}^{n} p_i^t$ and $p = \frac{1}{T}\sum_{t=1}^{T} p^t$.

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- By the definition of p, we have, for every player $i \in P$ and $a_i' \in A_i$,

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- The right-hand sides are time-averaged expected costs of i when playing according to the algorithm with small external regret and when playing a_i' every iteration. Since every player has regret at most ε, we obtain

$$\frac{1}{T}\sum_{t=1}^{T} \mathbb{E}_{a\sim p^t}[C_i(a)] \le \frac{1}{T}\sum_{t=1}^{T} \mathbb{E}_{a\sim p^t}[C_i(a_i'; a_{-i})] + \varepsilon.$$

- This verifies the ε-CCE condition for $p = \frac{1}{T}\sum_{t=1}^T p_t$.

Converging to CCE

Theorem 2.54

For every $G = (P, A, C)$, $\varepsilon > 0$, and $T = T(\varepsilon) \in \mathbb{N}$, if after T steps of the No-regret dynamics, each player $i \in P$ has time-averaged expected regret at most ε, then p is ε-CCE where $p^t = \prod_{i=1}^{n} p_i^t$ and $p = \frac{1}{T}\sum_{t=1}^{T} p^t$.

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- We modify a sequence $(p^t)_{t=1}^T$ with F by replacing it with a sequence $(f^t)_{t=1}^T$, where $f^t = (f_1^t, \ldots, f_N^t)$ and $f_i^t = \sum_{j\colon F(j)=i} p_j^t$.

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- Given a set of modification rules $\mathcal{F}$, we can compare our agent to his modifications by rules from $\mathcal{F}$, obtaining different notions of regret.

Internal and swap regret

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- Since $\mathcal{F}^{ex}, \mathcal{F}^{in} \subseteq \mathcal{F}^{sw}$, we immediately have $R^T_{A,\mathcal{F}^{ex}}, R^T_{A,\mathcal{F}^{in}} \leq R^T_{A,\mathcal{F}^{sw}}$.

The No-swap-regret dynamics

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Algorithm 0.14: NO-SWAP-REGRET DYNAMICS(G, T, ε)

Input : A normal-form game $G = (P, A, C)$ of n players, $T \in \mathbb{N}$, and $\varepsilon > 0$.
Output : A prob. distribution p_i^t on A_i for each $i \in P$ and $t \in \{1, \ldots, T\}$.
for every step $t = 1, \ldots, T$
do { Each player $i \in P$ independently chooses a mixed strategy p_i^t using an algorithm with average swap regret at most ε, with actions corresponding to pure strategies.
Each player $i \in P$ receives a loss vector $\ell_i^t = (\ell_i^t(a_i))_{a_i \in A_i}$, where $\ell_i^t(a_i) \leftarrow \mathbb{E}_{a_{-i}^t \sim p_{-i}^t}[C_i(a_i; a_{-i}^t)]$ for the product distribution $p_{-i}^t = \prod_{j \neq i} p_j^t$.
Output $\{p^t : t \in \{1, \ldots, T\}\}$.

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Algorithm 0.15: NO-SWAP-REGRET DYNAMICS(G, T, ε)

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- No-swap-regret dynamics then converges to a correlated equilibrium.


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