Algorithmic game theory

Martin Balko

6th lecture

November 11th 2025

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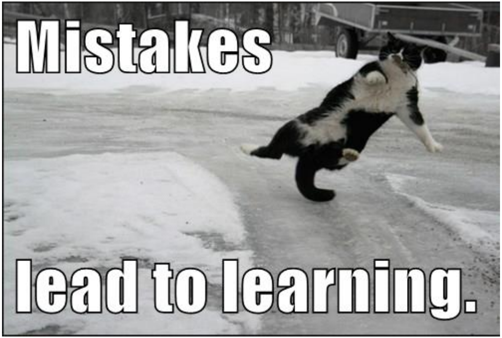

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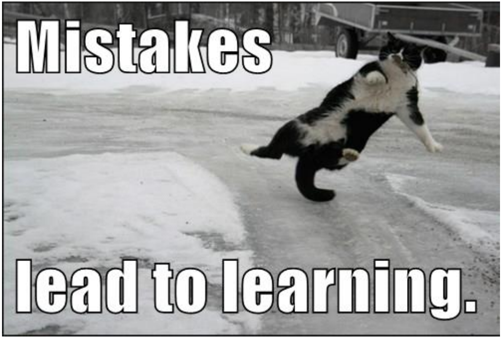

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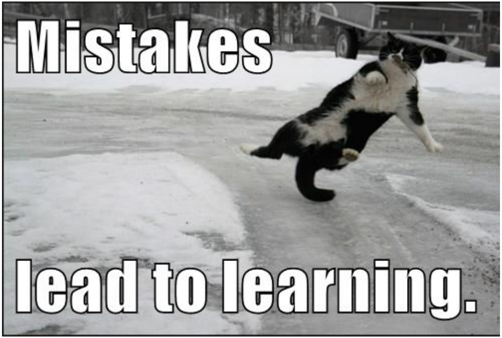

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- Today, we introduce the model and some basic algorithms on how to minimize regret.

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- Until specified otherwise, we consider only loss vectors from $\{0,1\}^N$. This is only to simplify the notation, all presented results can be extended to the general case.

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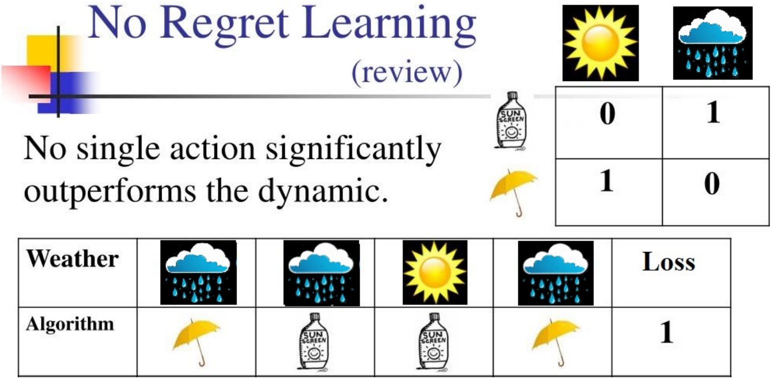
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(review)
SUN SCREEN
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0
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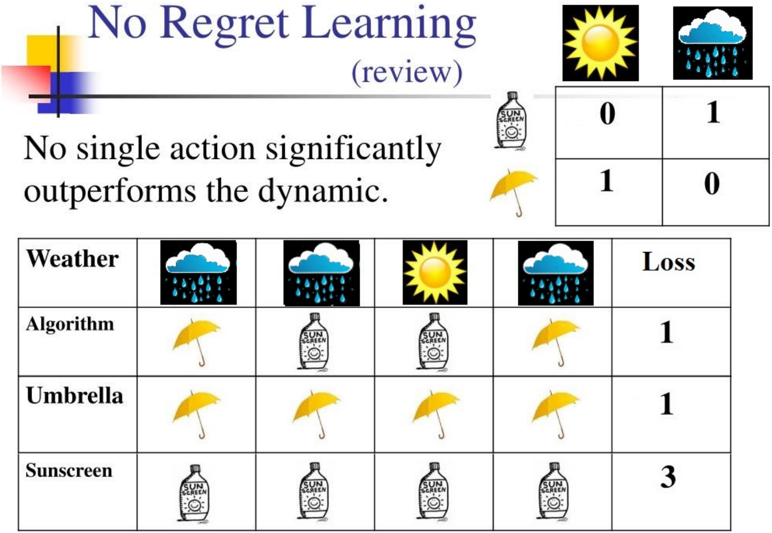
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For any agent A and every $T \in \mathbb{N}$, there is a sequence of T loss vectors and an agent $B \in \mathcal{A}_{all}$ such that $L_A^T - L_B^T \geq T(1 - 1/N)$.

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do $\begin{cases} L_{min}^{t-1} \leftarrow \min_{j \in X}\{L_j^{t-1}\}, \\ S^{t-1} \leftarrow \{i \in X \colon L_i^{t-1} = L_{min}^{t-1}\}, \\ k \leftarrow \min S^{t-1}, \end{cases}$

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Output $\{p^t \colon t \in \{1, \ldots, T\}\}$.

Analysis of the Greedy algorithm

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Proposition 2.46

For any sequence of $\{0,1\}$-valued loss vectors, the cumulative loss L^T_{Greedy} of the Greedy algorithm at time $T \in \mathbb{N}$ satisfies

$$L^T_{\text{Greedy}} \leq N \cdot L^T_{min} + (N-1).$$

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□

- This is rather weak since A can perform roughly N times worse than the best action.

Randomized Greedy algorithm

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Algorithm 0.19: RANDOMIZED GREEDY ALGORITHM(X, T)

Input : A set of actions $X = \{1, \ldots, N\}$ and number of steps $T \in \mathbb{N}$.
Output : A probability distribution p^t for every $t \in \{1, \ldots, T\}$.

Randomized Greedy algorithm

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- So it makes sense to introduce some randomness. We break ties at random, splitting weights between the currently best actions.

Algorithm 0.20: RANDOMIZED GREEDY ALGORITHM(X, T)

Input : A set of actions $X = \{1, \dots, N\}$ and number of steps $T \in \mathbb{N}$.
Output : A probability distribution p^t for every $t \in \{1, \dots, T\}$.
$p^1 \leftarrow (1/N, \dots, 1/N)$,

Randomized Greedy algorithm

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Algorithm 0.21: RANDOMIZED GREEDY ALGORITHM(X, T)

Input : A set of actions $X = \{1, \ldots, N\}$ and number of steps $T \in \mathbb{N}$.
Output : A probability distribution p^t for every $t \in \{1, \ldots, T\}$.
$p^1 \leftarrow (1/N, \ldots, 1/N)$,
for $t = 2, \ldots, T$
do $\{$

Randomized Greedy algorithm

- There is a good reason for the poor behavior. No deterministic algorithm can perform significantly better (Exercise).
- So it makes sense to introduce some randomness. We break ties at random, splitting weights between the currently best actions.

Algorithm 0.22: RANDOMIZED GREEDY ALGORITHM(X, T)

Input : A set of actions $X = \{1, \ldots, N\}$ and number of steps $T \in \mathbb{N}$.
Output : A probability distribution p^t for every $t \in \{1, \ldots, T\}$.
$p^1 \leftarrow (1/N, \ldots, 1/N)$,
for $t = 2, \ldots, T$
do $\left\{ L^{t-1}_{min} \leftarrow \min_{j \in X}\{L^{t-1}_j\}, \right.$

Randomized Greedy algorithm

- There is a good reason for the poor behavior. No deterministic algorithm can perform significantly better (Exercise).
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Algorithm 0.23: RANDOMIZED GREEDY ALGORITHM(X, T)

Input : A set of actions $X = \{1, \ldots, N\}$ and number of steps $T \in \mathbb{N}$.
Output : A probability distribution p^t for every $t \in \{1, \ldots, T\}$.
$p^1 \leftarrow (1/N, \ldots, 1/N)$,
for $t = 2, \ldots, T$

$$\textbf{do} \begin{cases} L_{min}^{t-1} \leftarrow \min_{j \in X}\{L_j^{t-1}\}, \\ S^{t-1} \leftarrow \{i \in X : L_i^{t-1} = L_{min}^{t-1}\}, \\ \end{cases}$$

Randomized Greedy algorithm

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Algorithm 0.24: RANDOMIZED GREEDY ALGORITHM(X, T)

Input : A set of actions $X = \{1, \ldots, N\}$ and number of steps $T \in \mathbb{N}$.
Output : A probability distribution p^t for every $t \in \{1, \ldots, T\}$.
$p^1 \leftarrow (1/N, \ldots, 1/N)$,
for $t = 2, \ldots, T$

$$\textbf{do} \begin{cases} L^{t-1}_{min} \leftarrow \min_{j \in X}\{L^{t-1}_j\}, \\ S^{t-1} \leftarrow \{i \in X \colon L^{t-1}_i = L^{t-1}_{min}\}, \\ p^t_i \leftarrow 1/|S^{t-1}| \text{ for every } i \in S^{t-1} \text{ and } p^t_i \leftarrow 0 \text{ otherwise.} \end{cases}$$

Randomized Greedy algorithm

- There is a good reason for the poor behavior. No deterministic algorithm can perform significantly better (Exercise).
- So it makes sense to introduce some randomness. We break ties at random, splitting weights between the currently best actions.

Algorithm 0.25: RANDOMIZED GREEDY ALGORITHM(X, T)

Input : A set of actions $X = \{1, \ldots, N\}$ and number of steps $T \in \mathbb{N}$.
Output : A probability distribution p^t for every $t \in \{1, \ldots, T\}$.
$p^1 \leftarrow (1/N, \ldots, 1/N)$,
for $t = 2, \ldots, T$
do $\begin{cases} L_{min}^{t-1} \leftarrow \min_{j \in X}\{L_j^{t-1}\}, \\ S^{t-1} \leftarrow \{i \in X \colon L_i^{t-1} = L_{min}^{t-1}\}, \\ p_i^t \leftarrow 1/|S^{t-1}| \text{ for every } i \in S^{t-1} \text{ and } p_i^t \leftarrow 0 \text{ otherwise.} \end{cases}$
Output $\{p^t \colon t \in \{1, \ldots, T\}\}$.

Analysis of the Randomized greedy algorithm

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Proposition 2.48

For any sequence of $\{0,1\}$-valued loss vectors, the cumulative loss L^T_{RG} of the Randomized greedy algorithm at time $T \in \mathbb{N}$ satisfies

$$L^T_{\mathrm{RG}} \leq (1 + \ln N) \cdot L^T_{min} + \ln N.$$

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For any sequence of $\{0,1\}$-valued loss vectors, the cumulative loss L_{RG}^{T} of the Randomized greedy algorithm at time $T \in \mathbb{N}$ satisfies

$$L_{\mathrm{RG}}^{T} \leq (1 + \ln N) \cdot L_{min}^{T} + \ln N.$$

- Proof (sketch): We proceed as in the previous proof. For $j \in \mathbb{N}$, let t_j be the time step t at which the loss L_{min}^{t} first reaches value j. We estimate the loss of the algorithm between steps t_j and t_{j+1}.

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- Note that $1 \leq |S^t| \leq N$.

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$$L_{\mathrm{RG}}^{T} \leq (1 + \ln N) \cdot L_{min}^{T} + (1/N + 1/(N-1) + \cdots + 1/(|S^T| + 1)).$$

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$$L_{\mathrm{RG}}^{T} \leq (1 + \ln N) \cdot L_{min}^{T} + (1/N + 1/(N-1) + \cdots + 1/(|S^T| + 1)).$$

$\square$

Polynomial weights algorithm

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Polynomial weights algorithm

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Algorithm 0.30: POLYNOMIAL WEIGHTS ALGORITHM(X, T, η)

Input : A set of actions $X = \{1, \ldots, N\}$, $T \in \mathbb{N}$, and $\eta \in (0, 1/2]$.
Output : A probability distribution p^t for every $t \in \{1, \ldots, T\}$.

Polynomial weights algorithm

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Algorithm 0.31: Polynomial weights algorithm(X, T, η)

Input : A set of actions $X = \{1, \ldots, N\}$, $T \in \mathbb{N}$, and $\eta \in (0, 1/2]$.
Output : A probability distribution p^t for every $t \in \{1, \ldots, T\}$.
$w_i^1 \leftarrow 1$ for every $i \in X$,

Polynomial weights algorithm

- This is better, but still not optimal. The losses are greatest when the sets S^t are small since the loss can be viewed as proportional to $1/|S^t|$.
- We overcome this issue by assigning larger weights to actions that are close to the best one.

Algorithm 0.32: Polynomial weights algorithm(X, T, η)

Input : A set of actions $X = \{1, \ldots, N\}$, $T \in \mathbb{N}$, and $\eta \in (0, 1/2]$.
Output : A probability distribution p^t for every $t \in \{1, \ldots, T\}$.
$w_i^1 \leftarrow 1$ for every $i \in X$,
$p^1 \leftarrow (1/N, \ldots, 1/N)$,

Polynomial weights algorithm

- This is better, but still not optimal. The losses are greatest when the sets S^t are small since the loss can be viewed as proportional to $1/|S^t|$.
- We overcome this issue by assigning larger weights to actions that are close to the best one.

Algorithm 0.33: POLYNOMIAL WEIGHTS ALGORITHM(X, T, η)

Input : A set of actions $X = \{1, \ldots, N\}$, $T \in \mathbb{N}$, and $\eta \in (0, 1/2]$.
Output : A probability distribution p^t for every $t \in \{1, \ldots, T\}$.
$w_i^1 \leftarrow 1$ for every $i \in X$,
$p^1 \leftarrow (1/N, \ldots, 1/N)$,
for $t = 2, \ldots, T$
do $\left\{ \right.$

Polynomial weights algorithm

- This is better, but still not optimal. The losses are greatest when the sets S^t are small since the loss can be viewed as proportional to $1/|S^t|$.
- We overcome this issue by assigning larger weights to actions that are close to the best one.

Algorithm 0.34: POLYNOMIAL WEIGHTS ALGORITHM(X, T, η)

Input : A set of actions $X = \{1, \ldots, N\}$, $T \in \mathbb{N}$, and $\eta \in (0, 1/2]$.
Output : A probability distribution p^t for every $t \in \{1, \ldots, T\}$.
$w_i^1 \leftarrow 1$ for every $i \in X$,
$p^1 \leftarrow (1/N, \ldots, 1/N)$,
for $t = 2, \ldots, T$
do $\left\{ w_i^t \leftarrow w_i^{t-1}(1 - \eta \ell_i^{t-1}), \right.$

Polynomial weights algorithm

- This is better, but still not optimal. The losses are greatest when the sets S^t are small since the loss can be viewed as proportional to $1/|S^t|$.
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$$\textbf{do} \begin{cases} w_i^t \leftarrow w_i^{t-1}(1 - \eta \ell_i^{t-1}), \\ W^t \leftarrow \sum_{i \in X} w_i^t, \\ \end{cases}$$

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Input : A set of actions $X = \{1, \ldots, N\}$, $T \in \mathbb{N}$, and $\eta \in (0, 1/2]$.
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Algorithm 0.37: Polynomial weights algorithm(X, T, η)

Input : A set of actions $X = \{1, \ldots, N\}$, $T \in \mathbb{N}$, and $\eta \in (0, 1/2]$.
Output : A probability distribution p^t for every $t \in \{1, \ldots, T\}$.
$w_i^1 \leftarrow 1$ for every $i \in X$,
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$$\textbf{do} \begin{cases} w_i^t \leftarrow w_i^{t-1}(1 - \eta \ell_i^{t-1}), \\ W^t \leftarrow \sum_{i \in X} w_i^t, \\ p_i^t \leftarrow w_i^t / W^t \text{ for every } i \in X. \end{cases}$$

Output $\{p^t : t \in \{1, \ldots, T\}\}$.

Analysis of the Polynomial weights algorithm I

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For $\eta \in (0, 1/2]$, every sequence of $[-1, 1]$-valued loss vectors, and every $k \in X$, the cumulative loss L^T_{PW} of the Polynomial weights algorithm satisfies

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For integers N and T with $T < \lfloor \log_2 N \rfloor$, there exists a stochastic generation of losses such that, for every online algorithm A, we have $\mathbb{E}[L^T_A] \geq T/2$ and yet $L^T_{min} = 0$.

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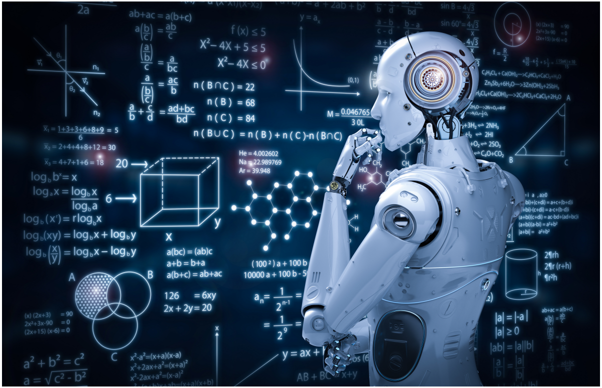
- See lecture notes for the proofs.
- We do not need to know T in advance (Exercise).

- Besides game theory, the “multiplicative weight update method” has many applications in various fields of science, for example in optimization, theoretical computer science, and machine learning.



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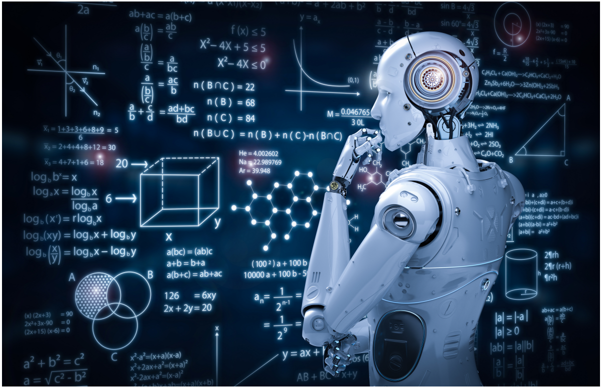

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- See `https://en.wikipedia.org/wiki/Multiplicative_weight_update_method#Applications`

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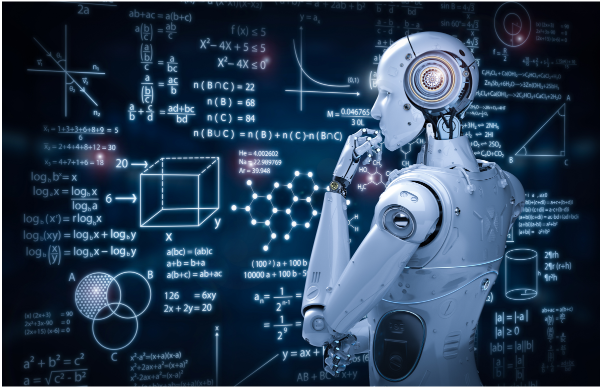

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- See `https://en.wikipedia.org/wiki/Multiplicative_weight_update_method#Applications`
- There are other algorithms producing small external regret, for example, the Regret matching algorithm.

Regret matching

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- Algorithm with external regret at most $O(\sqrt{T})$ for fixed N.

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$$p^t = \frac{[R^t]^+}{\|[R^t]^+\|_1} \quad \text{and} \quad R^{t+1} = R^t + \langle p^t, \ell^t \rangle \cdot \mathbf{1} - \ell^t,$$

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where $\mathbf{0}/0$ is defined as the uniform distribution.
- Strong empirical performance and admits many variants.

The No-regret dynamics

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The No-regret dynamics

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Algorithm 0.40: NO-REGRET DYNAMICS(G, T, ε)

Input : A normal-form game $G = (P, A, C)$ of n players, $T \in \mathbb{N}$, and $\varepsilon > 0$.
Output : A prob. distribution p_i^t on A_i for each $i \in P$ and $t \in \{1, \ldots, T\}$.
for every step $t = 1, \ldots, T$

do $\begin{cases} \text{Each player } i \in P \text{ independently chooses a mixed strategy } p_i^t \\ \text{using an algorithm with average regret at most } \varepsilon\text{, with actions} \\ \text{corresponding to pure strategies.} \\ \text{Each player } i \in P \text{ receives a loss vector } \ell_i^t = (\ell_i^t(a_i))_{a_i \in A_i}\text{, where} \\ \ell_i^t(a_i) \leftarrow \mathbb{E}_{a_{-i}^t \sim p_{-i}^t}[C_i(a_i; a_{-i}^t)] \text{ for the product distribution} \\ p_{-i}^t = \prod_{j \neq i} p_j^t. \end{cases}$

Output $\{p^t : t \in \{1, \ldots, T\}\}$.

"ENROLL IN AGT" THEY SAID

Algorithm 2.6.4: NO-REGRET DYNAMICS(G, T, ε)

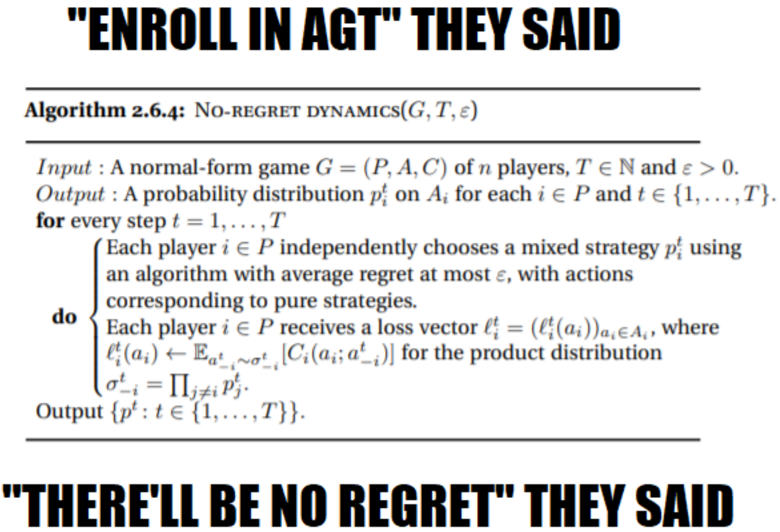
Input : A normal-form game $G = (P, A, C)$ of n players, $T \in \mathbb{N}$ and $\varepsilon > 0$.
Output : A probability distribution p_i^t on A_i for each $i \in P$ and $t \in \{1, \dots, T\}$.
for every step $t = 1, \dots, T$

do {
Each player $i \in P$ independently chooses a mixed strategy p_i^t using an algorithm with average regret at most ε, with actions corresponding to pure strategies.
Each player $i \in P$ receives a loss vector $\ell_i^t = (\ell_i^t(a_i))_{a_i \in A_i}$, where $\ell_i^t(a_i) \leftarrow \mathbb{E}_{a_{-i}^t \sim \sigma_{-i}^t}[C_i(a_i; a_{-i}^t)]$ for the product distribution $\sigma_{-i}^t = \prod_{j \neq i} p_j^t$.

Output $\{p^t : t \in \{1, \dots, T\}\}$.

"THERE'LL BE NO REGRET" THEY SAID

Sources: Students of MFF UK



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Thank you for your attention.