Algorithmic game theory

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12th lecture

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Proof of Myerson's lemma

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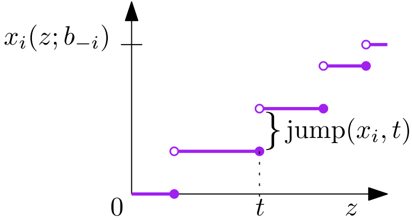

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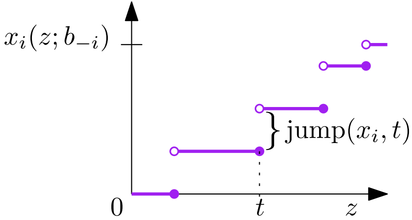

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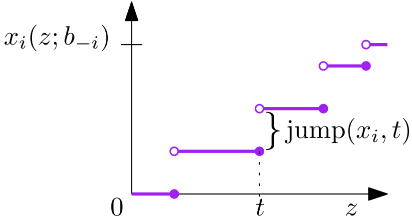

- For a piecewise constant function f, we use $\mathrm{jump}(f, t)$ to denote the magnitude of the jump of f at point t.
- If we fix z in the payment difference sandwich and let y approach z from below, then both sides become 0 if there is no jump of x_i at z. If $\mathrm{jump}(x_i, z) = h > 0$, then both sides tend to $z \cdot h$.

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- It will follow from a picture that it is optimal for bidder i to bid $b_i = v_i$.

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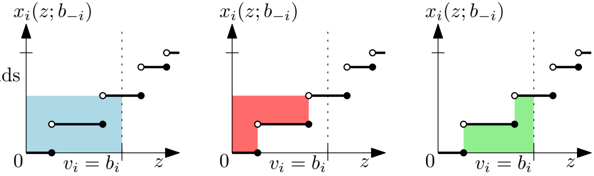
Bidder i bids truthfully:



(a) (b) (c)

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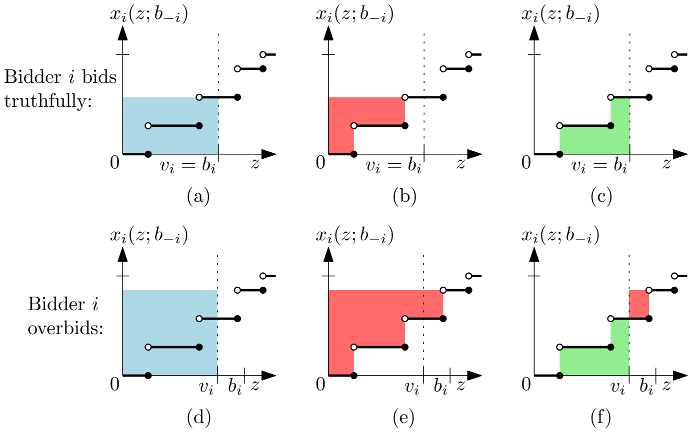

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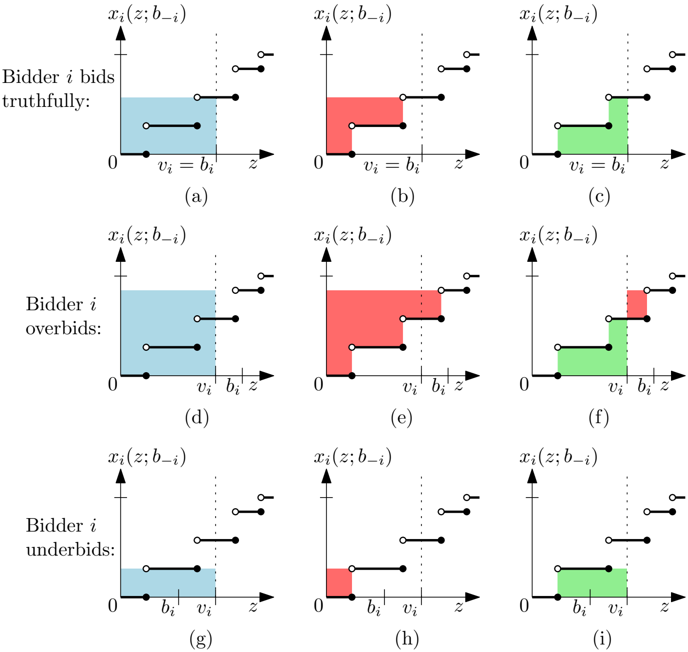

Knapsack auctions

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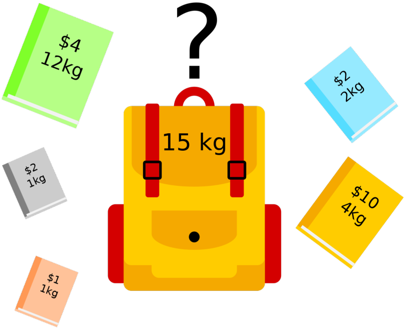

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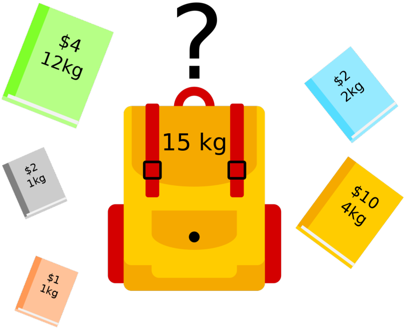

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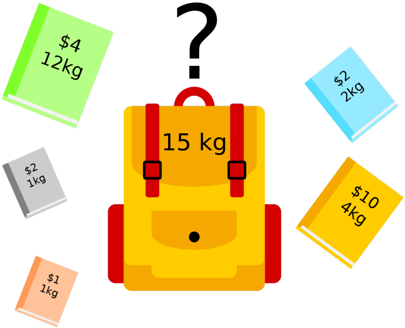

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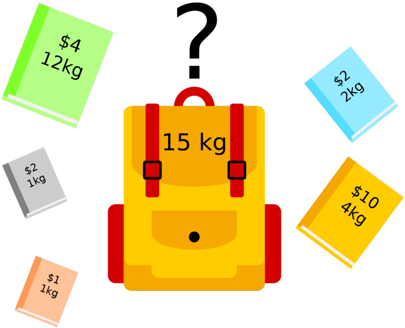

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- The dominant paradigm is to relax the second constraint (optimal surplus) as little as possible, subject to the first (DSIC) and the third (polynomial-time) constraints.
- Myerson's Lemma implies that the following goal is equivalent: design a polynomial-time and monotone allocation rule that comes as close as possible to maximizing the social surplus.

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- We now illustrate this approach by designing an allocation rule that gives at least half of the optimum social surplus in knapsack auctions.

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- The rule x_G is monotone (Exercise).

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- We show that this algorithm maximizes the surplus over all feasible solutions to the fractional knapsack problem.
  - Let $1, \dots, k$ be the winners selected by the greedy algorithm and suppose for contradiction that there is another feasible solution that gives higher social surplus.

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The Vickrey–Clarke–Groves (VCG) mechanism

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VCG mechanism (Theorem 3.18)

In every multi-parameter mechanism design environment, there is a DSIC social-surplus-maximizing mechanism.

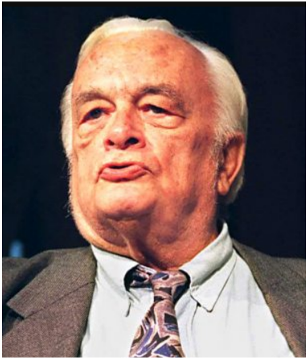

Figure: William Vickrey, Edward H. Clarke, and Theodore Groves.

Sources: : https://en.wikipedia.org, https://www.demandrevelation.com/, and https://www.researchate.net/

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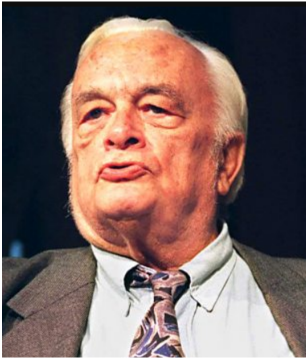

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- We now present the proof.

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where $\omega^* = x(b)$ is the outcome chosen by our allocation rule x for given bids b.

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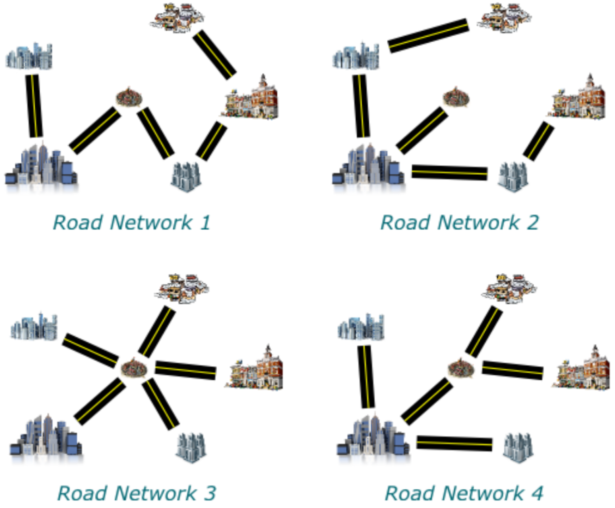

Sources: https://www.science4all.org/article/auction-design/

VCG auction example

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	Road Network 1	Road Network 2	Road Network 3	Road Network 4
	6 M$	14 M$	2 M$	**16 M$**
	5 M$	8 M$	4 M$	**12 M$**
	2 M$	1 M$	**20 M$**	4 M$
	4 M$	**6 M$**	3 M$	5 M$
	1 M$	1 M$	**6 M$**	2 M$
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- Cities pay their negative externalities on the collectivity. Other cities would be happier without the biggest city (NYC, say). How much happier they would be is exactly what NYC must pay.

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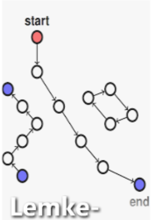
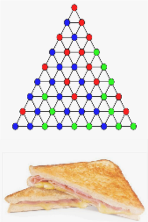
- If NYC was not there, then road network number 3 (RN3) would have been chosen, as opposed to RN4. The value of RN3 for the other cities would be 35 M$, as opposed to the 26 M$ of RN4. Therefore, the negative externality of NYC is $35 - 26 = 9$ M$.

Nash equilibria

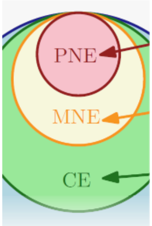
Minimax theorem



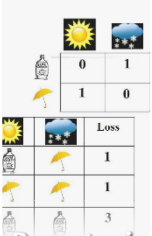
Lemke-Howson algorithm



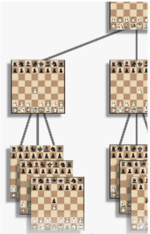
Complexity of NASH



Variants of NE



Regret minimization

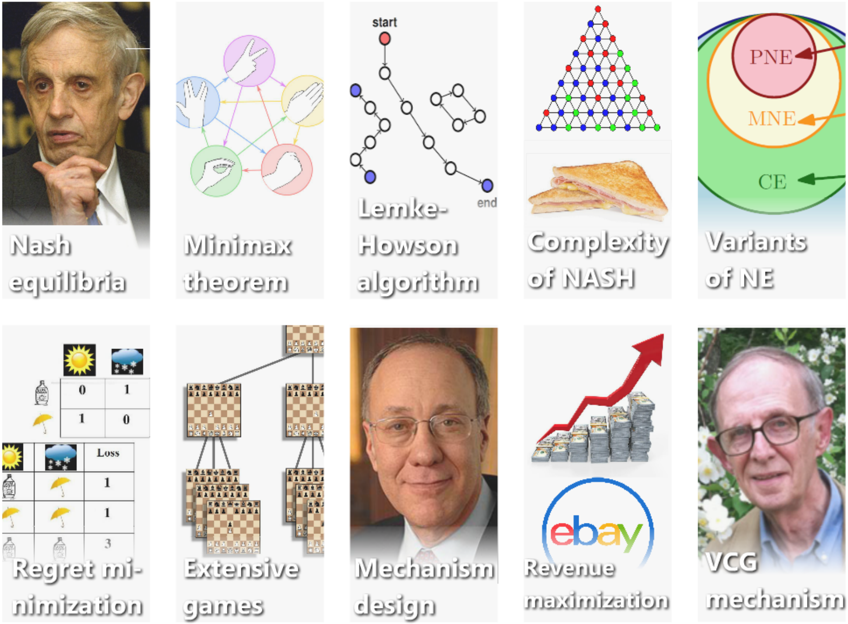


Extensive games

Mechanism design

Revenue maximization

VCG mechanism



Thank you for your attention.

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- The key idea turns is considering the the loss of social surplus inflicted on the other $n - 1$ bidders by the presence of bidder i.

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