

Algorithmic game theory

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What we learned last time

- For **single-item auctions** we have found an **awesome auction** (Vickrey's **auction**) with the following three properties:
 - **DSIC**: everybody has dominant strategy “bid truthfully” which guarantees non-negative utility,
 - **strong performance**: maximizing social surplus,
 - **computational efficiency**: running in polynomial time.
- We then generalized single-item auctions to **single-parameter environments**.
 - 1 seller and n bidders, the seller collects the bids $b = (b_1, \dots, b_n)$,
 - the seller chooses an allocation $x(b) = (x_1(b), \dots, x_n(b))$ from X ,
 - the seller sets payments $p(b) = (p_1(b), \dots, p_n(b))$.
- Is there awesome mechanism (x, p) for single-parameter environments?
- We started by looking for **DSIC** mechanisms and saw a powerful tool for designing them.

Myerson's lemma

- An allocation rule x is **implementable** if there is a payment rule p such that (x, p) is DSIC.
- An allocation rule x is **monotone** if, for every bidder i and all bids b_{-i} , the allocation $x_i(z; b_{-i})$ is nondecreasing in z .

Myerson's lemma

In a single-parameter environment, the following three claims hold.

- (a) An allocation rule is **implementable if and only if it is monotone**.
- (b) If an allocation rule x is monotone, then there exists a **unique payment rule** p such that (x, p) is DSIC (assuming $b_i = 0$ implies $p_i(b) = 0$).
- (c) For every i , the payment rule p is given by the following **explicit formula**

$$p_i(b_i; b_{-i}) = \int_0^{b_i} z \cdot \frac{d}{dz} x_i(z; b_{-i}) dz.$$

- We saw two applications: **single-item** and **sponsored-search auctions**.

Social surplus maximization

- So far, we were designing DSIC mechanisms for single-parameter environments that maximize the **social surplus** $\sum_{i=1}^n v_i x_i(b)$.
- We did not care about the **revenue** $\sum_{i=1}^n p_i(b)$.
- This makes sense in some real-world scenarios where revenue is not the first-order objective. For instance, in government auctions (e.g., to sell wireless spectrum).



Source: youtube.com

- In **single-item auctions**, maximizing social surplus is done by **Vickrey's auction**. In general single-parameter environments, we use **Myerson's lemma**.
- Today, we try to maximize the revenue.

Revenue maximizing auctions



Source: Reprofoto

Revenue maximization

- How to maximize the **revenue** $\sum_{i=1}^n p_i(b)$?



Source: <https://merger.com/recurring-revenue/>

- The situation then becomes more complicated, but we will see some nice results today.

Why is it more complicated?

- Consider single-item auction with 1 bidder with valuation v .



Source: <https://auctions.propertyolvers.co.uk/>

- the **only DSIC auction**: the seller posts a price r , then his revenue is either r if $v \geq r$ and 0 otherwise.
- Maximizing the **social surplus** is trivial by putting $r = 0$.
- However, when maximizing the **revenue**, it is not clear how we should set r , since we do not know the valuation v .

New model

- Thus, we need a model to reason about trade-offs between our performance on different inputs.
- We introduce some randomness outside of discrete probability spaces, so let us first recall some notions from the **probability theory**.
 - If F is a probability distribution drawing some value v , then $F(z)$ is the probability that v has value at most z .
 - If F is a probability distribution with **density** f and with support $[0, v_{max}]$, then

$$f(x) = \frac{d}{dx} F(x) \text{ and } F(x) = \int_0^x f(z) dz.$$

- For a random variable X , the **expected value of X** is

$$\mathbb{E}_{z \sim F}[X(z)] = \int_0^{v_{max}} X(z) \cdot f(z) dz.$$

Bayesian model

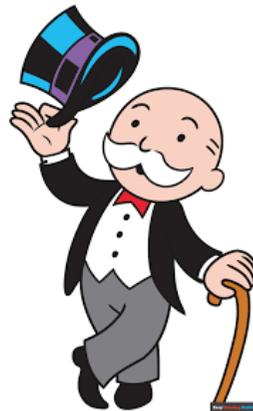
- The **Bayesian model** consists of
 - a single-parameter environment (x, p) ,
 - for each bidder i , the private valuation v_i of i is drawn from a probability distribution F_i with density function f_i and with support contained in $[0, v_{max}]$. We assume that the distributions F_1, \dots, F_n are independent, but not necessarily the same.
- The distributions F_1, \dots, F_n are known to the mechanism designer. The bidders do not know the distributions F_1, \dots, F_n . Since we discuss only DSIC auctions, the bidders do not need to know F_1, \dots, F_n , as they have dominant strategies.
- The goal is to maximize, among all DSIC auctions, the **expected revenue**

$$\mathbb{E}_{v=(v_1, \dots, v_n) \sim (F_1 \times \dots \times F_n)} \left[\sum_{i=1}^n p_i(v) \right],$$

where the expectation is taken with respect to the given distribution $F_1 \times \dots \times F_n$ over the valuations (v_1, \dots, v_n) .

Example: single-bidder single-item auction

- Consider the example with a **single bidder in a single-item auction**. It is now easier to handle.
- The expected revenue of a posted price r equals $r \cdot (1 - F_1(r))$, since $(1 - F_1(r))$ is the probability that v is at least r .
- Given a probability distribution F_1 , it is usually a simple matter to maximize this value.
- For example, if F_1 is the **uniform distribution** on $[0, 1]$, where $F_1(x) = x$ for every $x \in [0, 1]$, then we achieve the maximum revenue by choosing $r = 1/2$.
- The posted price r that makes the expected revenue as high as possible is called the **monopoly price**.



Example: two-bidders single-item auction

- Consider a single-item auction with **two bidders** that have their valuations v_1 and v_2 drawn **uniformly** from $[0, 1]$.



- Now, there are more DSIC auctions to think of, for example, the **Vickrey's auction**, for which the expected revenue is $1/3$ (**Exercise**).
- Besides Vickrey's auctions, there are **other DSIC auctions that perform better** with respect to the expected revenue (**Exercise**).
- Today, we describe such optimal auctions in more general settings using **Myerson's lemma**.

Maximizing expected revenue

- To characterize DSIC auctions that maximize the expected revenue, we find a **more useful formula for the expected revenue**.

Theorem 3.13

Let (x, p) be a DSIC mechanism in a single-parameter environment forming a Bayesian model with n bidders, probability distributions F_1, \dots, F_n , and density functions f_1, \dots, f_n . Let $F = F_1 \times \dots \times F_n$. Then

$$\mathbb{E}_{v \sim F} \left[\sum_{i=1}^n p_i(v) \right] = \mathbb{E}_{v \sim F} \left[\sum_{i=1}^n \varphi_i(v_i) \cdot x_i(v) \right],$$

where

$$\varphi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

is the **virtual valuation** of bidder i .

Maximizing expected revenue: remarks

- In other words, for every DSIC auction, the expected revenue is equal to the expected **virtual social surplus** $\sum_{i=1}^n \varphi_i(v_i) \cdot x_i(v)$. So when maximizing the expected revenue, we end up with maximizing a similar value as before.
- Note that the virtual valuation $\varphi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ of a bidder can be negative and that it depends on v_i and F_i , and not on parameters of the others.
- **Example**: consider a bidder i with valuation v_i drawn from the uniform distribution on $[0, 1]$. Then $F_i(z) = z$, $f_i(z) = 1$, and $\varphi_i(z) = z - (1 - z)/1 = 2z - 1 \in [-1, 1]$.
- A **coarse intuition** behind the virtual valuations: the first term v_i in this expression can be thought of as the maximum revenue obtainable from bidder i and the second term as the inevitable revenue loss caused by not knowing v_i in advance.

Maximizing expected revenue: proof I

- Since (x, p) is DSIC, we assume $b_i = v_i$ for each bidder i .
- By [Myerson's lemma](#), we have the following payment formula

$$p_i(b_i; b_{-i}) = \int_0^{b_i} z \cdot \frac{d}{dz} x_i(z; b_{-i}) dz.$$

- It can be shown that this formula holds for **any monotone function** $x_i(z; b_{-i})$, if we suitably interpret the derivation $\frac{d}{dz} x_i(z; b_{-i})$. All arguments in this proof, such as integration by parts, can be made rigorous for bounded monotone functions, but we omit the details.
- We fix a bidder i and the bids v_{-i} of others. By the payment formula, the expected payment of i for v_{-i} is determined by x_i and equals

$$\begin{aligned} \mathbb{E}_{v_i \sim F_i} [p_i(v)] &= \int_0^{v_{\max}} p_i(v) f_i(v_i) dv_i \\ &= \int_0^{v_{\max}} \left(\int_0^{v_i} z \cdot \frac{d}{dz} x_i(z; v_{-i}) dz \right) f_i(v_i) dv_i. \end{aligned}$$

Maximizing expected revenue: proof II

- After reversing the integration order (**Fubini's theorem**), we obtain

$$\int_0^{v_{\max}} \left(\int_0^{v_i} z \cdot \frac{dx_i(z; v_{-i})}{dz} dz \right) f_i(v_i) dv_i = \int_0^{v_{\max}} \left(\int_z^{v_{\max}} f_i(v_i) dv_i \right) z \cdot \frac{dx_i(z; v_{-i})}{dz} dz.$$

- We have used the fact that $z \leq v_i$. Since f_i is the density function, the inner integral can be simplified and we get

$$\int_0^{v_{\max}} (1 - F_i(z)) \cdot z \cdot \frac{dx_i(z; v_{-i})}{dz} dz.$$

- We apply **integration by parts**, that is $\int fg' = fg - \int f'g$, choosing $f(z) = (1 - F_i(z)) \cdot z$ and $g'(z) = \frac{d}{dz} x_i(z; v_{-i})$, we get

$$[(1 - F_i(z)) \cdot z \cdot x_i(z; v_{-i})]_0^{v_{\max}} - \int_0^{v_{\max}} x_i(z; v_{-i}) \cdot (1 - F_i(z) - zf_i(z)) dz.$$

- The **first term is zero**, as it equals to 0 for $z = 0$ and also for $z = v_{\max}$, as v_{\max} is a finite number.

Maximizing expected revenue: proof III

- Thus, so far we derived that the expected revenue of i for v_{-i} is

$$0 - \int_0^{v_{max}} x_i(z; v_{-i}) \cdot (1 - F_i(z) - zf_i(z)) dz.$$

- By the **definition of $\varphi_i(v_i)$** , the second term can be rewritten as

$$\int_0^{v_{max}} \left(z - \frac{1 - F_i(z)}{f_i(z)} \right) x_i(z; v_{-i}) f_i(z) dz = \int_0^{v_{max}} \varphi_i(z) x_i(z; v_{-i}) f_i(z) dz.$$

- This is an expected value where $z \sim F_i$ and thus, for every v_{-i} ,

$$\mathbb{E}_{v_i \sim F_i} [p_i(v_i; v_{-i})] = \mathbb{E}_{v_i \sim F_i} [\varphi_i(v_i) \cdot x_i(v_i; v_{-i})].$$

- Taking the expectation with respect to v_{-i} , this is true also for $v \sim F$.
- Finally, applying the **linearity of expectation** twice, we get the final form

$$\mathbb{E}_{v \sim F} \left[\sum_{i=1}^n p_i(v) \right] = \sum_{i=1}^n \mathbb{E}_{v \sim F} [p_i(v)] = \sum_{i=1}^n \mathbb{E}_{v \sim F} [\varphi_i(v_i) x_i(v)] = \mathbb{E}_{v \sim F} \left[\sum_{i=1}^n \varphi_i(v_i) x_i(v) \right]$$

□

Maximizing expected virtual social surplus

- By [Theorem 3.13](#), “maximizing expected revenue = maximizing expected virtual social surplus”. We characterize auctions that do that.
- We start with [single-item auctions](#). We assume that there is a probability distribution F such that $F_1 = \dots = F_n = F$ (thus all virtual valuations $\varphi_1, \dots, \varphi_n$ are equal to a function φ). We also assume that F is [regular](#), that is, $\varphi(v)$ is strictly increasing in v .
- All this holds if F is the uniform probability distribution $F(v) = v$ on $[0, 1]$, as then $\varphi(v) = v - (1 - F(v))/f(v) = 2v - 1$.
- We show that for such auctions the Vickrey auction with reserved price maximizes expected revenue.
- In a [Vickrey auction with reserve price \$r\$](#) , the item is given to the highest bidder, unless all bids are less than r , in which case no one gets the item. The winner (if any) is charged the second-highest bid or r , whichever is larger.

Vickrey with reserve price is optimal

Proposition 3.14

Let F be a regular probability distribution with density f and the virtual valuation φ and let F_1, \dots, F_n be independent probability distributions on valuations of n bidders such that $F = F_1 = \dots = F_n$ (and thus $\varphi = \varphi_1 = \dots = \varphi_n$). Then, **Vickrey auction with reserve price $\varphi^{-1}(0)$ maximizes the expected revenue.**

- **Proof:** By **Theorem 3.13**, we want to maximize the expected virtual social surplus. To do so, we can choose $x(b) \in X$ for each input b , while we do not have any control over F and φ .
- In a single-item auction, the feasibility constraint is $\sum_{i=1}^n x_i(b) \leq 1$, so we give the item to bidder i with the highest $\varphi(b_i)$. If every bidder has a negative virtual valuation, then we do not award the item to anyone.
- It remains to show that this allocation rule x is monotone, as then, by **Myerson's lemma**, it can be extended to a DSIC auction (x, p) . Since F is regular, the function φ is strictly increasing. Thus, x is monotone and we get Vickrey's auction with a reserve price $\varphi^{-1}(0)$. □

- The Vickrey auction with reserve price is used by eBay.

Address <http://cgi.ebay.com/ws/eBayISAPI.dll?ViewItem&item=300244882874>

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FREE shipping

Current bid: **US \$38.37**

Your maximum bid: **US \$** **Place Bid >**
(Enter US \$39.37 or more)

End time: **Aug-03-08 20:28:12 PDT (6 days)**

Shipping costs: **Free**
US Postal Service First Class Mail®
Service to [United States](#)
([more services](#))

Ships to: **Worldwide**

Item location: **Marion County, South Carolina, United States**

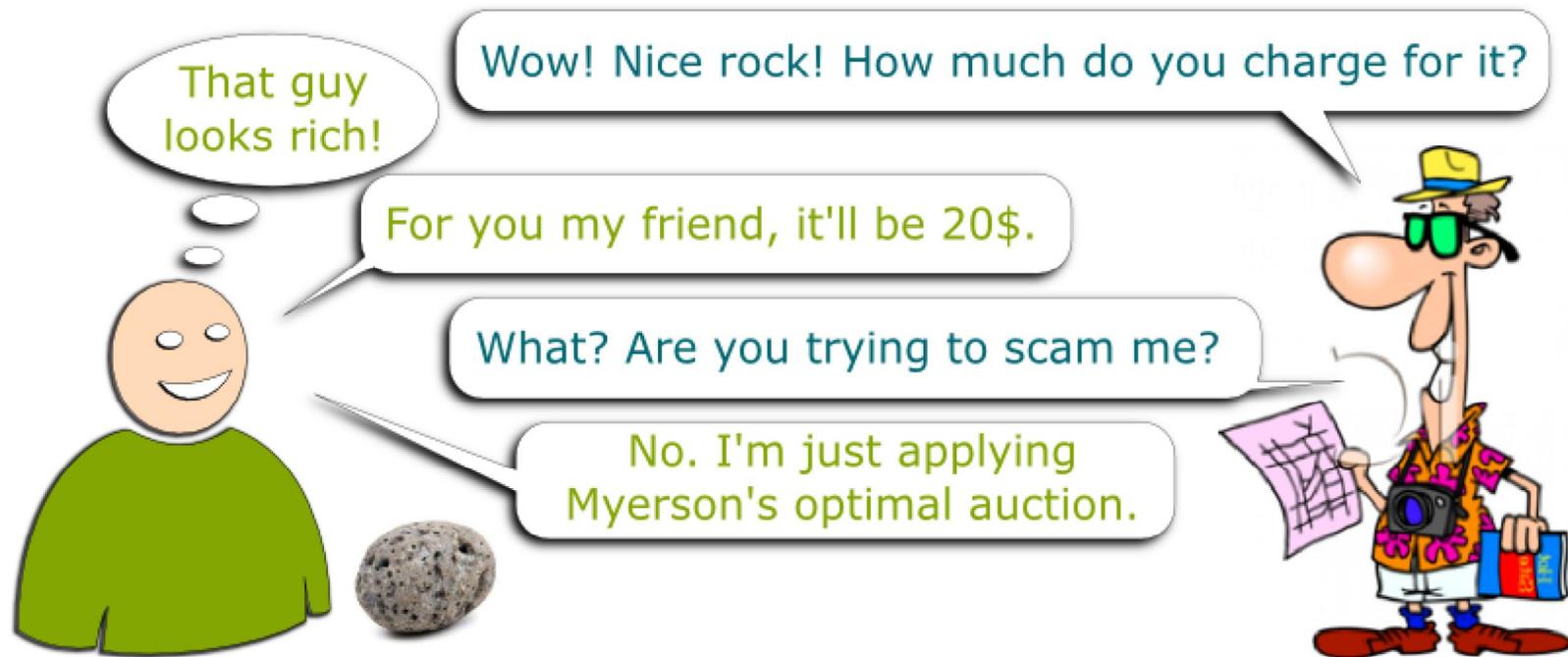
History: [24 bids](#)

High bidder: [Q***h](#) (92 ★)

[View larger picture](#)

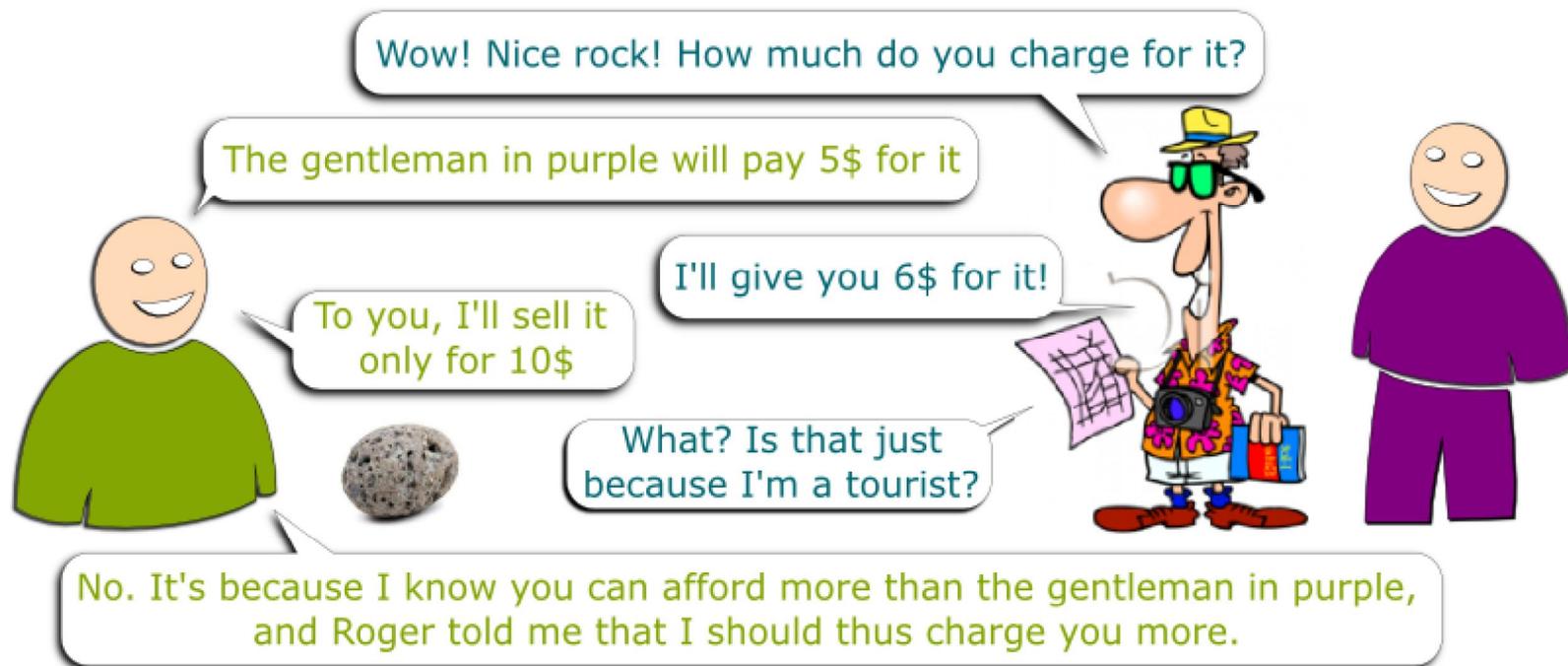
Reserve price and maximizing revenue

- The theory was developed by [Roger Myerson](#).
- The result can be extended to single-parameter environments. With more work we can prove a version for distributions that are not regular.
- Very roughly, if the seller believes that bidders have high valuations, he should set a high reserve price accordingly.

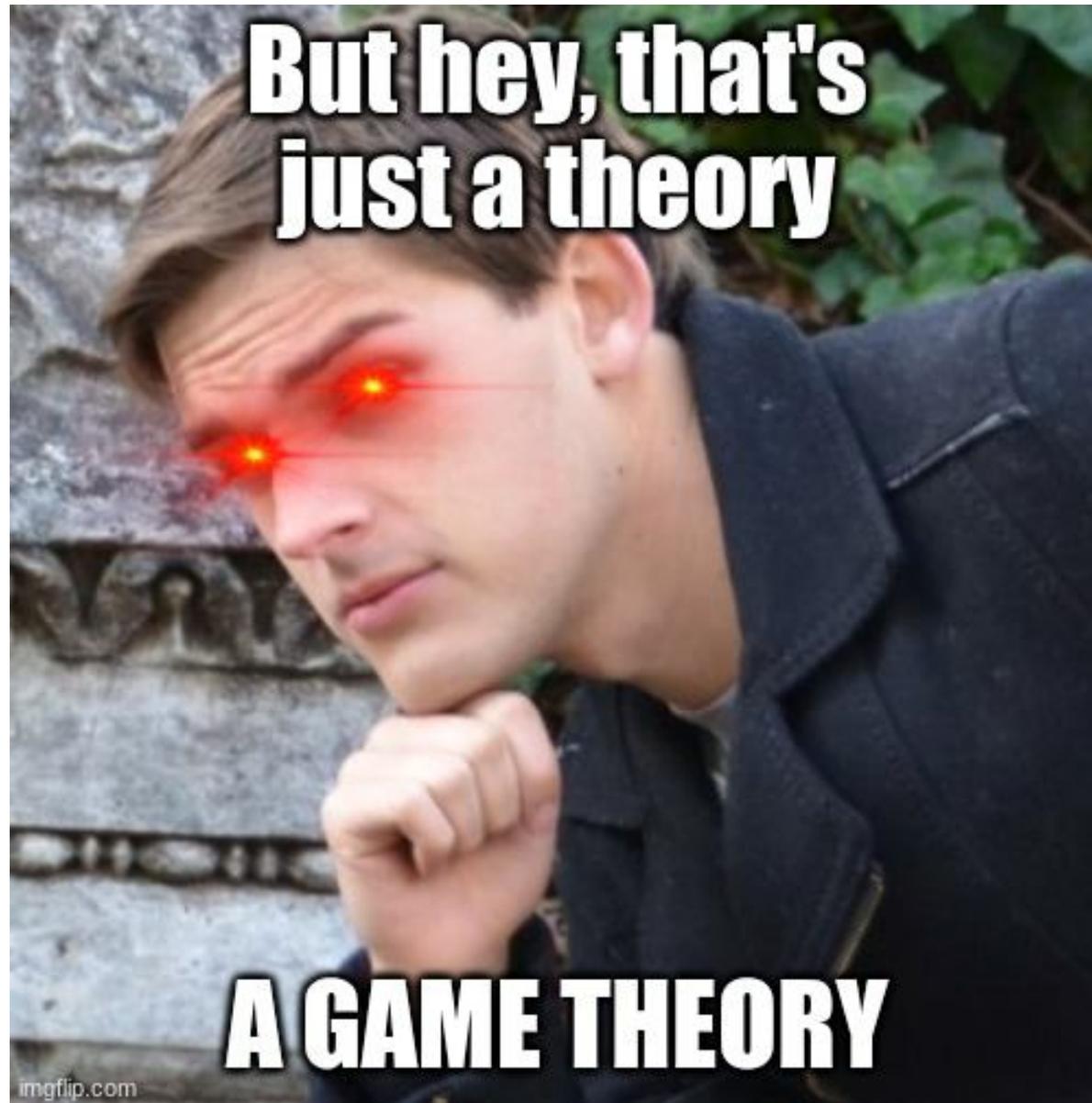


Optimal auctions more generally

- There are also optimal DISC auctions even if we relax the conditions by not insisting on F_1, \dots, F_n being identical. However, such optimal auctions can get weird, and they do not generally resemble any auctions used in practice (**Exercise**).



Source: <https://www.science4all.org/article/auction-design/>



Source: <https://www.reddit.com/>

Exams

Exams info

- **Exam format:**
 - Oral exam with preparation, 3 hours max.
 - I will ask about **three topics**, one survey question, one question with proofs, and one exercise.
 - The more points you have, the easier the exam is. If you have at least 85 points, you do not need to solve the exercise part.
- **Dates (so far):**
 - **19.12.** – 15:00–18:00, capacity 10
 - **2.1.** – 13:00–16:00, capacity 10
 - **12.1.** – 9:00–19:00, capacity 30
 - **15.1.** – 9:00–19:00, capacity 30
 - **5.2.** – 9:00–19:00, capacity 30
 - **19.2.** – 9:00–19:00, capacity 30
 - **12.2.** – 9:00–19:00, capacity 30
- **What you should know:** everything that we covered (everything is included in the lecture notes).



Source: <https://www.kindpng.com/>

Thank you for your attention and merry Christmas!