Algorithmic game theory – Tutorial 1*

October 10, 2019

1 Nash equilibria

A *normal-form game* is a triple (P, A, u), where P is a finite set of n *players*, $A = A_1 \times \cdots \times A_n$ is a set of *action profiles*, where A_i is a set of *actions* available to player i, and $u = (u_1, \dots, u_n)$ is an n-tuple, where each $u_i \colon A \to \mathbb{R}$ is the *utility function* for player i.

The set of *pure strategies* of player i is the set A_i of available actions for i. The set S_i of *mixed strategies* of player i is the set of all probability distributions on A_i. The *expected payoff* for player i of the mixed-strategy profile $s = (s_1, \dots, s_n)$ is

$$u_i(s) = \sum_{a=(a_1,\dots,a_n)\in A} u_i(a) \prod_{j=1}^{n} s_j(a_j).$$

We use the notation $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$ and, for a strategy $s'_i \in S_i$ of player i, we use $u_i(s'_i; s_{-i})$ to denote the number $u_i(s_1, \dots, s_{i-1}, s'_i, s_{i+1}, \dots, s_n)$.

The *best response* of player i to the strategy profile s_{-i} is a mixed strategy s^*_i such that $u_i(s^*_i; s_{-i}) \geq u_i(s'_i; s_{-i})$ for each strategy $s'_i \in S_i$ of i. A *Nash equilibrium* in G is a strategy profile $(s_1, \dots, s_n)$ such that s_i is a best response of player i to s_{-i} for every $i \in P$.

Exercise 1. *Verify that the expected payoff of a mixed strategy in a normal-form game $G = (P, A, u)$ of n players is linear. That is, prove that $u_i(s) = \sum_{a_i \in A_i} s_i(a_i) u_i(a_i; s_{-i})$ for every player $i \in P$ and every mixed-strategy profile $s = (s_1, \dots, s_n)$.*

Exercise 2. *Compute mixed Nash equilibria in the following games:*

(a) *Prisoner's dilemma,*

(b) *Rock-Paper-Scissors.*

and formally show that no other Nash equilibria exist in these games.

Exercise 3 (Iterated dominance equilibrium)**.** *Let $G = (P, A, u)$ be a normal-form game of n players. For player i, we say that a strategy $s_i \in S_i$ is* strictly dominated *by a strategy $s'_i \in S_i$ if, for every $s_{-i} \in S_{-i}$, we have $u_i(s_i; s_{-i}) < u_i(s'_i; s_{-i})$. Consider the following iterated process that will help us find Nash equilibria in some games.*

Set $A^0_i = A_i$ and $S^0_i = S_i$ for every player $i \in P$. For $t \geq 1$ and $i \in P$, let A^t_i be the set of pure strategies from A^{t-1}_i that are not strictly dominated by a strategy from S^{t-1}_i and let S^t_i be the set of mixed strategies with support contained in A^t_i. Let T be the first step, when the sets A^T_i and S^T_i are no longer shrinking for any $i \in P$. If each player $i \in P$ is left with one strategy $a_i \in A^T_i$, we call $a_1 \times \cdots \times a_n$ an iterated dominance equilibrium *of G.*

(a) *Show that every iterated dominance equilibrium is a Nash equilibrium.*

(b) *Find an example of a two-person game in normal form game with a Nash equilibrium that is not iterated dominance equilibrium.*

Exercise 4. *Use iterated elimination of strictly dominated strategies (introduced in Exercise 3) to find the unique Nash equilibrium in the following normal-form game of 2 players (see Table 1) by first reducing the game to a 2×2 game.*

Exercise 5. *Consider the following game of $n \geq 2$ players. Every player selects, independently, a number from $\{1, \dots, 1000\}$. The goal of each player is to have his number closest to the half of the average of all the selected numbers.*

We define two variants of this game, depending on the tie-breaking rule. In the first rule, all players that are closest to the half of the average split evenly the payoff of 1. In the second tie-breaking rule, each player that is closest to the half of the average receives payoff of 1.

How would you play each of these games? Find all pure Nash equilibria of the game under

*Information about the course can be found at http://kam.mff.cuni.cz/~balko/

	c_1	c_2	c_3	c_4
r_1	(5, 2)	(22, 4)	(4, 9)	(7, 6)
r_2	(16, 4)	(18, 5)	(1, 10)	(10, 2)
r_3	(15, 12)	(16, 9)	(18, 10)	(11, 3)
r_4	(9, 15)	(23, 9)	(11, 5)	(5, 13)

Table 1: A game from Exercise 4.

(a) *the first tie-breaking rule,*

(b) *the second tie-breaking rule.*