On the Empty Hexagon Theorem

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Abstract

Tobias Gerken has very recently solved a well-known open problem of Erdős by showing that there is an integer c with the following property. If P is a finite set of at least c points in general position in the plane then there is a convex hexagon with all vertices lying in P and with no point of P lying inside the hexagon. We give a proof of this result that is directed somewhat differently than Gerken's proof. We also give a simple algorithm that finds an empty hexagon in a given point set in the optimal linear time.

1 Introduction

Let X be a finite set of points in the plane. We say that X is *in general position*, if no three points of X lie on a line. The convex hull of X is denoted by conv$X$. We say that X is *in convex position*, if each point of X is a vertex of conv$X$. The interior of conv$X$ is denoted by int$X$.

A classical result in discrete geometry is the following theorem:

Theorem 1 (Erdős–Szekeres Theorem [5]) *For every $k \geq 3$ there is a (smallest) integer* ES(k) *such that any set of at least* ES(k) *points in general position in the plane contains k points in convex position.*

Let P be a finite set of points in general position in the plane. A convex k-gon G is called a *k-hole* (or *empty convex k-gon) of P*, if all vertices of G lie in P and no point of P lies inside G.

Erdős [4] asked if, for a fixed k, any sufficiently large point set has a k-hole. Already many years ago, this was known to be true for $k \leq 5$ [7] and false for $k \geq 7$ [8]. The remaining case $k = 6$ became a well-known open problem. Gerken [6] has very recently solved it in the affirmative:

Theorem 2 (The Empty Hexagon Theorem [6]) *There is an integer c such that any set of at least c points in general position in the plane has a 6-hole.*

In this paper we give a proof of Theorem 2 that is directed somewhat differently than the proof of Gerken (see also Paragraph 4.1 for remarks on Gerken's proof). Our proof gives another view at the structure of point set with no empty hexagons (or with a small number of empty hexagons).

Dobkin *et al.* [3] gave algorithms for finding empty triangles, for finding empty r-gons ($r > 3$), and for determining a largest empty convex subset. Based on Theorem 2, in the next section we give a simple algorithm that finds an empty hexagon in a given planar point set in the optimal linear time.

Further research related to Theorems 1 and 2 is described in several survey papers [1, 2, 9, 11].

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2 The algorithm

Every point of the input set must be visited at least once, since otherwise we couldn't exclude that a convex hexagon found by the algorithm contains some of the unvisited points in its interior. This gives a linear lower bound.

Let P be a set of $n \geq c$ points in general position in the plane, where c is the constant from Theorem 2. We may assume that the x-coordinates of the points in P are pairwise different. The idea of the algorithm is to find a subset $Q \subset P$ of size from $[c, 2c]$ such that the convex hull of Q contains no other points of P and then to find an empty hexagon of Q in time $O(1)$.

If P has size at most $2c$ then an empty hexagon can be found in time $O(1)$ using, for example, the algorithm of [3]. We further suppose that the size of P is bigger than $2c$. We take arbitrary $2c+1$ points of P and find the median, m, among their x-coordinates. This can done in time $O(1)$. We then compare the x-coordinate of each point in P with m. Let P_1 contain the points $p \in P$ with $x(p) < m$ and let P_2 contain the points $p \in P$ with $x(p) > m$. Let P' be the smaller of the sets P_1, P_2. We then repeat the above process for the set P' (of size satisfying $c \leq |P'| < |P|/2$). The process is repeated at most $\log_2 n$ times. Thus, an empty hexagon is found in time $O(\log n \cdot 1) + O(n + n/2 + n/4 + \cdots) + O(1) = O(n)$.

3 Proof of Theorem 2

3.1 Outline of the proof

Let P be a finite set of points in general position in the plane, containing no 6-hole. Let $k \geq 3$. A convex k-gon $\mathrm{conv}X$, $X \subseteq P$, is *minimal (for P)*, if no k-gon $\mathrm{conv}Y$, $Y \subset P$, satisfies $\mathrm{conv}Y \subset \mathrm{conv}X$ and $\mathrm{conv}Y \neq \mathrm{conv}X$.

Let A be the vertex set of a minimal convex polygon for P. We will prove Theorem 2 by showing that $|A|$ is bounded by an absolute constant.

Let B be the vertex set of $\mathrm{conv}(P \cap \mathrm{int}A)$ (naturally, if $X \subseteq P, |X| \leq 2$, then X is taken for the vertex set of $\mathrm{conv}X$). Similarly, let C be the vertex set of $\mathrm{conv}(P \cap \mathrm{int}B)$ and let D be the vertex set of $\mathrm{conv}(P \cap \mathrm{int}C)$ (see Figure 1).

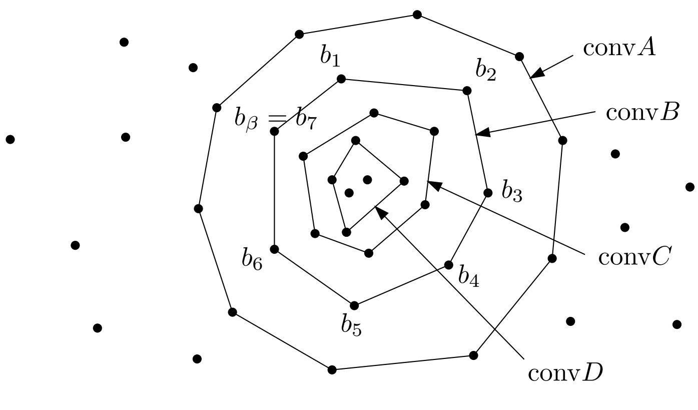

Figure 1: The "layers" A, B, C, D.

Here is our key lemma:

Lemma 1 *If $|A| \geq 7$ then $D = \emptyset$.*

Before proving Lemma 1, we show that it easily implies Theorem 2 with $c = \mathrm{ES}(216)$[1]. Let $|P| \geq \mathrm{ES}(216)$, and let A be the vertex set of any minimal 216-gon. Lemma 1 gives $D = \emptyset$. It follows that $|C| \leq 5$, since otherwise any six vertices of C would form a 6-hole. The convex hull of any six consecutive vertices of conv$B$ contains a point of C, since otherwise it would be a 6-hole. It follows that $|B| \leq 6|C| + 5 \leq 35$. Analogously $|A| \leq 6|B| + 5 \leq 215$. This contradicts $|A| = 216$.

It remains to prove Lemma 1.

3.2 Two observations

In the proof of Lemma 1 we often use the two observations below.

Observation 1 *The only points of P lying in the interior of the region* conv$A \setminus$ conv$D$ *are the points of B and C.* □

If $p_1p_2 \ldots p_k$ is a convex k-gon ($k \geq 3$), then we define a *k-sector* $S(p_1, p_2, \ldots, p_k)$ as the region of points $q \in \mathbf{R}^2$ such that $qp_1p_2 \ldots p_k$ is a convex $(k+1)$-gon (see Figure 2). Note that any k-sector is either a triangle or a region bounded by a segment and two half-lines.

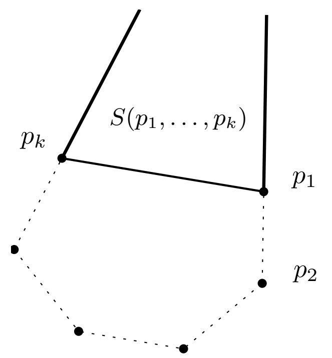

Figure 2: A k-sector.

Observation 2 *If a convex pentagon $p_1p_2p_3p_4p_5$ with vertices in P is empty and $q \in P \cap S(p_1, p_2, p_4, p_5)$, then P contains a 6-hole.*

Proof. If the triangle qp_1p_5 is empty then $qp_1p_2p_3p_4p_5$ is a 6-hole. Otherwise $rp_1p_2p_3p_4p_5$ is a 6-hole for some $r \in P$ inside the triangle qp_1p_5. □

3.3 Proof of Lemma 1

Set $\alpha := |A|, \beta := |B|$. We will show that if $\alpha \geq 7$ and $D \neq \emptyset$ then either P contains a 6-hole or conv$A$ is not minimal.

Suppose $D \neq \emptyset$ and fix an arbitrary point $d \in D$. Denote the vertices of B in the clockwise order by $b_1, b_2, \ldots, b_\beta$ (see Figure 1). For $i = 1, \ldots, \beta$, let T_i be the triangle db_ib_{i+1} (see Figure 3; we always identify the indices in a natural way so that $b_i = b_{\beta+i}$ for any integer i).

Let $i \in \{1, \ldots, \beta\}$. If T_i contains a point of C, then we fix a $c_i \in C \cap T_i$ such that the triangle $c_ib_ib_{i+1}$ is empty (by Observation 1, we may choose c_i as the point of $C \cap T_i$ closest to the line b_ib_{i+1}). Otherwise c_i is not defined.

[1]We improve the constant relatively easily to $c = \mathrm{ES}(15)$ in Section 4. Gerken [6] gave a more complicated proof of Theorem 2 with $c = \mathrm{ES}(9)$. See also further comments in Section 4.

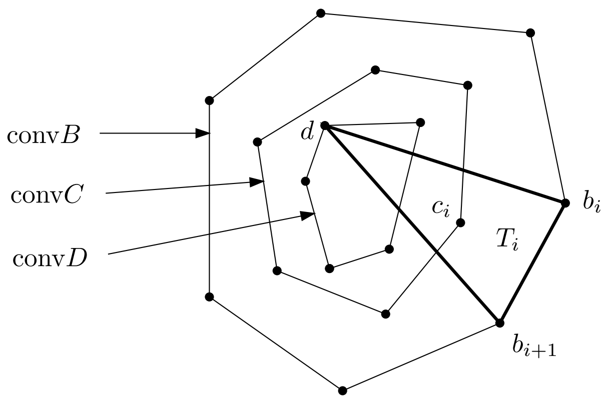

Figure 3: The triangle T_i and the point c_i.

Observation 3 *For each i, $T_i \cup T_{i+1}$ contains a point of C. Thus, at least one of c_i, c_{i+1} is defined.*

Proof. Suppose that $T_i \cup T_{i+1}$ contains no point of C. Since d lies inside conv$C$, the region $T_i \cup T_{i+1}$ is intersected by a unique edge cc' of conv$C$. By Observation 1, the pentagon $cb_ib_{i+1}b_{i+2}c'$ is empty (see the left picture in Figure 4). We may apply

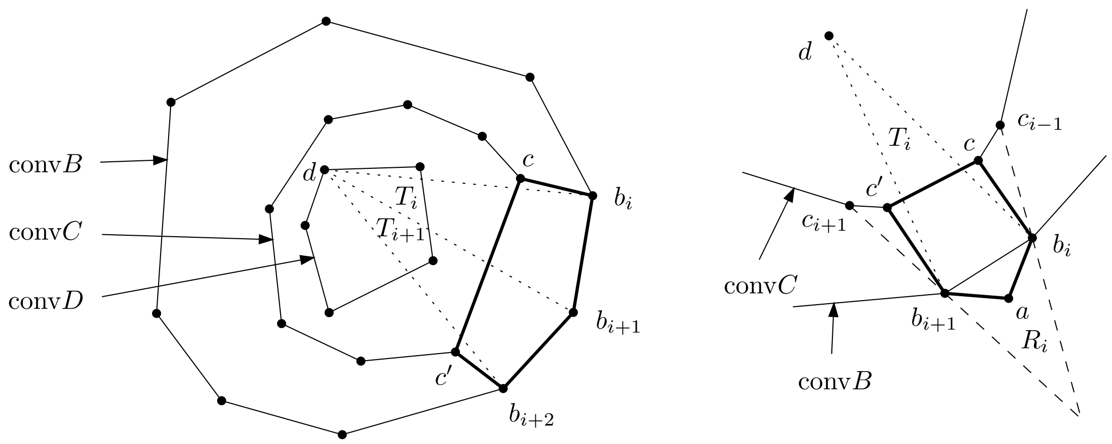

Figure 4: The empty pentagons constructed in the proofs of Observations 3 (left) and 4 (right).

Observation 2 on this pentagon and on d. It implies the existence of a 6-hole — a contradiction. □

Let $i \in \{1, \ldots, \beta\}$. If c_i is defined, then we define $S_i := S(b_i, c_i, b_{i+1})$ (see Figure 5). Otherwise c_{i-1}, c_{i+1} are defined according to Observation 3, and we define $R_i := S(b_i, c_{i-1}, c_{i+1}, b_{i+1})$.

Observe that the exterior of conv$B$ is covered by the 3-sectors $S(b_i, d, b_{i+1})$, $i = 1, \ldots, \beta$. Each $S(b_i, d, b_{i+1})$ is covered either by S_i (if S_i is defined) or by $S_{i-1} \cup R_i \cup S_{i+1}$ (otherwise). Thus, the β sectors S_i and R_i cover the entire exterior

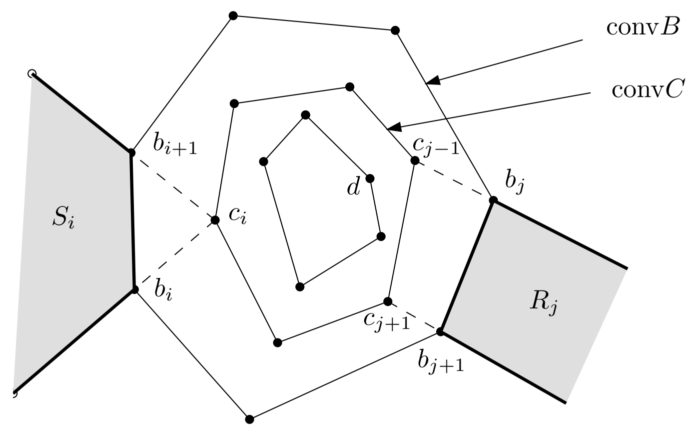

Figure 5: A 3-sector S_i and a 4-sector R_j.

of conv$B$. In particular, each point of A lies in at least one of them. In fact, the R_i's contain no points of A:

Observation 4 *Each R_i contains no point of A.*

Proof. Suppose a point $a \in A$ lies in R_i. A unique edge cc' of conv$C$ intersects T_i (possibly $c = c_{i-1}$ and/or $c' = c_{i+1}$). The convex pentagon $cb_iab_{i+1}c'$ is empty by Observation 1 (see the right picture in Figure 4). This pentagon and the point d satisfy the assumptions of Observation 2. Thus, P contains a 6-hole — a contradiction. □

Thus, all points of A lie in the union of the S_i's. Let $t \geq 1$. Whenever t consecutive 3-sectors $S_i, S_{i+1}, \ldots, S_{i+t-1}$ are defined, we denote their union by $S_{i,t}$:

$$S_{i,t} := S_i \cup S_{i+1} \cup \cdots \cup S_{i+t-1}.$$

The crucial tool in our proof is the following lemma, which restates and slightly generalizes Lemma 4 of [6].

Lemma 2 *If $S_{i,t}$ is defined and $t < \beta$, then $S_{i,t}$ contains at most $t+1$ points of A.*

Proof. We proceed by induction on t. Without loss of generality, we suppose that $i = 1$.

If $t = 1$, then $S_{1,1} = S_1$ contains at most two points of A since otherwise the points b_1, c_1, b_2 and any three points of $A \cap S_1$ form a 6-hole according to Observation 1.

Suppose now that $t > 1$ and that the lemma holds for $t-1$. We partition $S_{1,t}$ into $S_{2,t-1}$ and $W := S_{1,t} \setminus S_{2,t-1}$ (see the left picture in Figure 6). Note that $S_{2,t-1}$ contains at most t points by the inductive assumption.

Suppose that the quadrilateral $Q := b_1c_1c_2b_2$ is convex. Then W does not contain two different points $a, a' \in A$, since otherwise the 5-hole $c_1b_1aa'b_2$ and the point c_2 satisfy the assumptions of Observation 2. Therefore $S_{1,t} = W \cup S_{2,t-1}$ contains at most $1+t$ points of A in this case.

Thus, we may suppose that the quadrilateral Q is non-convex. This means that the point c_1 lies inside the triangle $b_1b_2c_2$. Analogously we may suppose on the "opposite" side of $S_{1,t}$ that the point c_t lies inside the triangle $b_tb_{t+1}c_{t-1}$. Then the polygon $H := b_1c_1c_2\ldots c_tb_{t+1}$ is convex. Let A' be the set in convex position

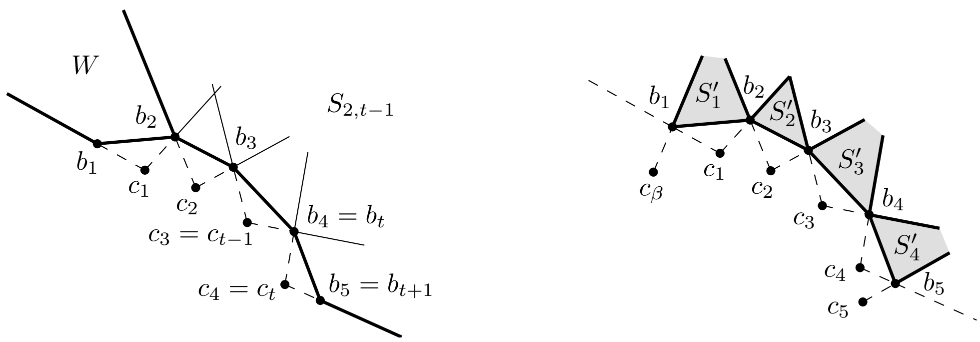

Figure 6: The partition of $S_{1,t}$ into $S_{2,t-1}$ and W (left) and four of the regions S'_i (right).

obtained from A by replacing the points of $A \cap S_{1,t}$ by the $t+2$ vertices of H. Observe that $S_{1,t} = S_1 \cup S_{2,t-1}$ contains at most $2+t$ points of A by the inductive assumption. Thus $|A'| \geq \alpha$ — a contradiction with the minimality of conv$A$. □

We now derive Lemma 1 from Lemma 2.

Suppose first that there is a $d \in D$ such that at least one of the triangles $T_i = db_ib_{i+1}$ contains no point of C. Let $i_1 < i_2 < \cdots < i_s$ be the indices $i \in \{1, \ldots, \beta\}$ for which the triangle T_i contains no point of C. For each $j = 1, \ldots, s-1$, the union of the 3-sectors S_i with $i_j < i < i_{j+1}$ contains at most $i_{j+1} - i_j$ points of A according to Lemma 2. Similarly the union of the remaining S_i's with $i > i_s$ and $i < i_1$ contains at most $(\beta + i_1) - i_s$ points of A. It follows that the union of all S_i's contains at most $(i_2 - i_1) + (i_3 - i_2) + \cdots + (i_s - i_{s-1}) + ((\beta + i_1) - i_s) = \beta$ points of A. Since the union of all S_i's contains all points of A, we obtain $\alpha \leq \beta$ — a contradiction with the minimality of conv$A$.

Thus we may suppose in the rest of the proof that the following condition holds:

(C) For any $d \in D$ and $i \in \{1, \ldots, \beta\}$, the triangle $T_i = db_ib_{i+1}$ contains at least one point of C.

Fix any $d \in D$. Note that under condition (C), c_i and S_i are defined for each $i = 1, \ldots, \beta$. Recall that the set $S := S_1 \cup S_2 \cup \cdots \cup S_\beta$ contains the α points of A. Since $S = S_1 \cup S_{2,\beta-1}$, an application of Lemma 2 on $S_1 = S_{1,1}$ and $S_{2,\beta-1}$ therefore gives $\alpha \leq 2+\beta$. On the other hand, the minimality of conv$A$ implies that $\alpha \geq \beta + 1$. We distinguish the two possible cases $\alpha = \beta + 1$ and $\alpha = \beta + 2$.

First suppose that $\alpha = \beta + 2$. For each i, set $S'_i := S \setminus S_{i+1,\beta-1}$ (see the right picture in Figure 6). Each S'_i contains at least two points of A, since otherwise Lemma 2 would imply that $S = S_{i+1,\beta-1} \cup S'_i$ contains at most $\beta+1 = \alpha-1$ points of A. Since the β regions S'_i are pairwise disjoint, A contains at least $\beta \cdot 2 \geq \beta+3 = \alpha+1$ points — a contradiction.

It remains to settle the case $\alpha = \beta + 1$. We can argue similarly as above that each of the β pairwise disjoint regions S'_i contains at least one point of A. For each $i = 1, \ldots, \beta$, fix a point $a_i \in A$ in S'_i. Since $\alpha = \beta + 1$, there is a unique point $a \in A$ different from $a_1, \ldots, a_\beta$. Without loss of generality, we may suppose that a lies between a_β and a_1 in the clockwise order along the boundary of conv$A$. For each i, the intersection of S_i with the boundary of A is connected and contains a_i and no $a_j, j \neq i$. It follows that a does not lie in S_i for any $i \neq 1, \beta$. Since we suppose that $\alpha \geq 7$, a does not lie in $S_{2,5} = S_2 \cup S_3 \cup \cdots \cup S_6$. It follows that the points of A lying in $S_{2,5}$ are exactly the five points $a_2, a_3, \ldots, a_6$.

If d lies inside the convex pentagon $G = c_2c_3c_4c_5c_6$ then the convex α-gon $a_1b_2c_2dc_6b_7a_7a_8\ldots a_\beta a$ contradicts the minimality of conv$A$ (see Figure 7).

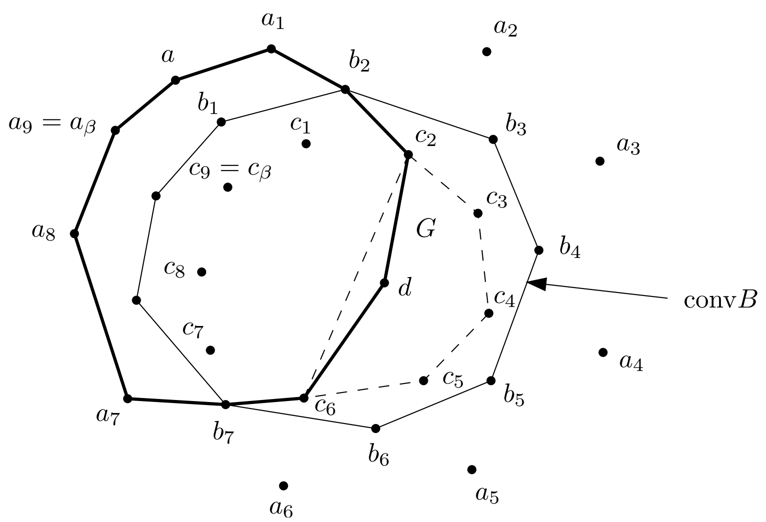

Figure 7: The convex α-gon $a_1b_2c_2dc_6b_7a_7a_8\ldots a_\beta a$ (for $\beta = 9$).

Suppose now that d does not lie in the pentagon G. The pentagon G is non-empty, since otherwise Observation 2 applied on G and d would imply the existence of a 6-hole. It follows that there is a $d' \in D$ inside G. Note that $C = \{c_1, \ldots, c_\beta\}$, since the existence of other points in C would imply that $|C| \geq \beta + 1 = \alpha$ — a contradiction with the minimality of conv$A$.

By (C) and by $|C| = \beta$, each triangle $T'_i = d'b_ib_{i+1}$ contains exactly one point of C. The cyclic order of the points of C in which they appear in the triangles $T'_1, T'_2, \ldots, T'_\beta$ equals the cyclic order in which they appear along the convex hull of C, which is the cyclic order in which they appear in the triangles $T_1, T_2, \ldots, T_\beta$. Thus, there is an integer $\Delta \in \mathbf{Z}$ such that T'_i contains $c_{i+\Delta}$ for each i. Now, let j be the unique index with $d \in T'_j$. Then $T_j \subset T'_j$ and therefore T'_j contains c_j. It follows that $\Delta = 0$. Therefore, we may replace d by d' and argue as above.

This concludes the proof of Lemma 1. Theorem 2 is proved.

4 Concluding remarks

4.1 Gerken's proof

Gerken [6] proves Theorem 2 by showing that a 6-hole appears in every minimal convex 9-gon A. He distinguishes, in our notation, various cases according to the sizes of the sets B, C, D. Gerken [6] achieves a reasonable bound $c \leq \mathrm{ES}(9) \leq 1717$ by considering the possible cases for $|A| = 9$. Instead of this approach, we have presented a general argument working for any sufficiently large $|A|$. Our approach in the case $D \neq \emptyset$ resembles the approach in one of the cases considered in [6]. A disadvantage of our proof is that it gives a worse bound on c than Gerken's proof.

4.2 A bound on the second innermost layer

The following lemma of independent interest is used in next paragraph for obtaining a better constant in Theorem 2:

Lemma 3 *Let $X \cup Y$ be a finite set of points in general position in the plane with no 6-hole. Suppose that X, Y are in convex position and each point of Y lies inside* conv$X$. *Then $|Y| \leq 5$ and $|X| \leq 7$. (Thus, if $D = \emptyset$ in the proof of Theorem 2 then $|C| \leq 5$ and $|B| \leq 7$.)*

Proof. The bound on $|Y|$ is obvious. If $Y = \emptyset$ then $|X| \leq 5$ and the lemma holds. If $1 \leq |Y| \leq 2$ then we define an auxiliary line l as follows. If $|Y| = 1$ then l is chosen as a line connecting the point of Y with an arbitrary point of X. If $|Y| = 2$ then l is the line through the two points of Y. In either case, each side of l contains at most three points of X. It follows that $|X| \leq 2 \cdot 3 + 1 = 7$.

Now, let $|Y| \geq 3$. Denote the vertices of Y in the clockwise order by $y_1, y_2, \ldots, y_{|Y|}$. For $|Y| = 3$, let $s_1, s_2, s_3, t_1, t_2, t_3$ be the numbers of points of X lying in the six regions depicted in the left picture in Figure 8. Then

$$\begin{array}{rcl} s_i + t_i + s_{i+1} & \leq & 3 \quad (i = 1, 2, 3), \\ t_i & \leq & 2 \quad (i = 1, 2, 3). \end{array}$$

If we sup up these six inequalities and divide the resulting inequality by two, we obtain $\sum s_i + \sum t_i \leq 7.5$, which implies $|X| = \sum s_i + \sum t_i \leq 7$.

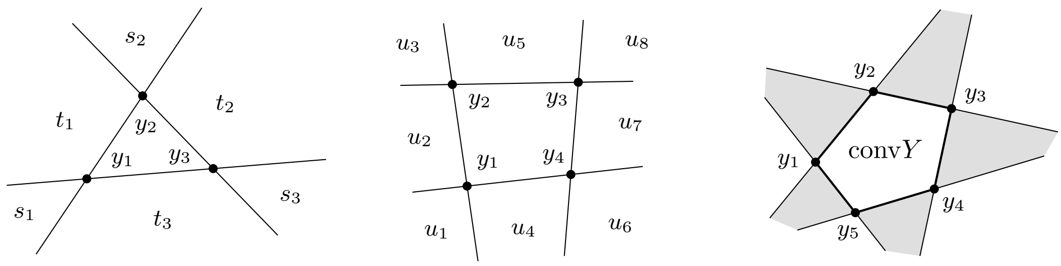

Figure 8: The regions considered in the proof of Lemma 3.

If $|Y| = 4$ then similarly (see the middle picture in Figure 8) $|X| = (u_1 + u_2 + u_3) + (u_4 + u_5) + (u_6 + u_7 + u_8) \leq 3 + 1 + 3 = 7$ in this case.

If $|Y| = 5$ then the five shaded 4-sectors $S(y_{i-1}, y_i, y_{i+1}, y_{i+2})$ in the right picture in Figure 8 contain no point of X (by Observation 2). Therefore each point of X is separated from conv$Y$ by at least two of the five lines $y_i y_{i+1}$. If $|X| > 7$ was satisfied, then some line $y_i y_{i+1}$ would separate at least $\lceil (|X| \cdot 2)/5 \rceil \geq \lceil (8 \cdot 2)/5 \rceil = 4$ points of X and any such four points and the corresponding points y_i, y_{i+1} would form a 6-hole — a contradiction. □

4.3 Better bounds in Theorem 2

Here we refine the argument given in Paragraph 3.1. Let P, A, B, C, D be as in Paragraph 3.1. We will prove Theorem 2 with $c = \mathrm{ES}(15)$ by showing that $|A| < 15$.

We may suppose that $D = \emptyset$, since otherwise $|A| \leq 6$ by Lemma 1. If $C = \emptyset$ then $|A| \leq 7$ by Lemma 3. Otherwise fix an arbitrary $c \in C$. Lemma 3 implies $\beta \leq 7$. Each of the β 3-sectors $S(b_i, c, b_{i+1})$ contains at most two points of A. Thus, A contains at most $\beta \cdot 2 \leq 14$ points. Since any set of at least ES(15) points in general position in the plane contains a (minimal) convex 15-gon, Theorem 2 with $c = \mathrm{ES}(15)$ follows.

The bound $c = \mathrm{ES}(15)$ may be further improved relatively easily (using a modification of Lemma 2, for example). However, we do not see how to achieve Gerken's constant $c = \mathrm{ES}(9)$ without an extensive case analysis as in [6]. A lower bound

$c \geq 30$ follows from Overmars' construction [10] of 29 points in general position with no 6-hole.

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