Helly numbers and topological complexity

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joint works with

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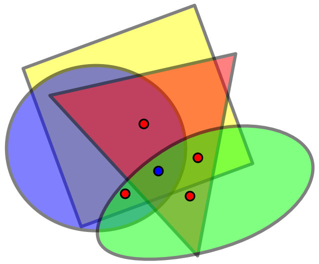
Pavel Patak (Charles University)
Zuzana Safernova (Charles University)
Martin Tancer (IST Vienna)
Uli Wagner (IST Vienna)

Helly's Theorem. Any finite family of convex sets in $\mathbb{R}^{d}$ has non-empty intersection if any $d+1$ elements have non-empty intersection.



Classical result in convex geometry

Related to Radon and Caratheodory's theorems...

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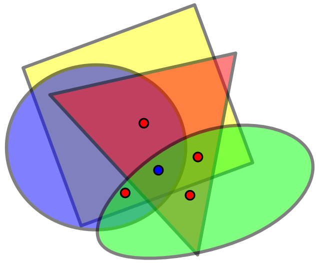

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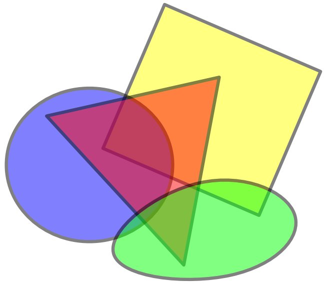
In the contrapositive:

> If finitely many convex sets in $\mathbb{R}^d$ have empty intersection, some d of them have empty intersection.



Statement about size of witnesses for empty intersection

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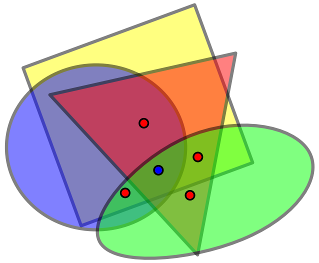

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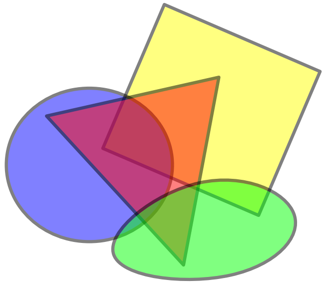
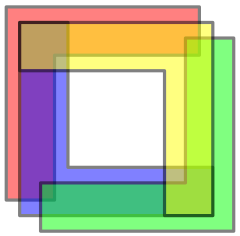


Statement about size of witnesses for empty intersection

Family-based rather than class-based formulation:

The Helly number of a family $\mathcal{F}$ of sets is the maximum size of an inclusion-minimum sub-family of $\mathcal{F}$ with empty intersection.



We implicitly assume that $\mathcal{F}$ has empty intersection

$$\text{Helly}(\mathcal{F}) = \max\{|\mathcal{G}| : \mathcal{G} \subseteq \mathcal{F}, \cap\mathcal{G} = \emptyset, \forall A \in \mathcal{G}, \cap(\mathcal{G} \setminus \{A\}) \neq \emptyset\}$$

Helly's Theorem. If $\mathcal{F}$ is a finite family of convex sets in $\mathbb{R}^d$ then $\text{Helly}(\mathcal{F}) \leq d+1$.

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Helly numbers arise naturally e.g. in optimization:

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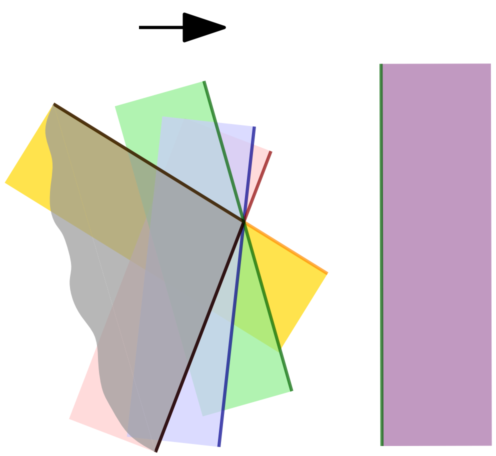
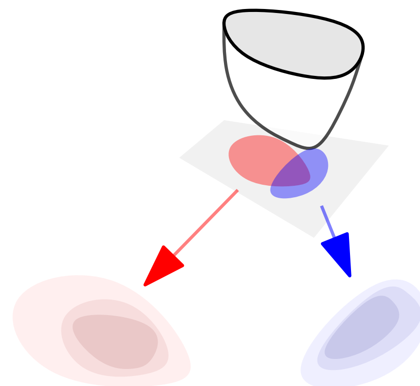
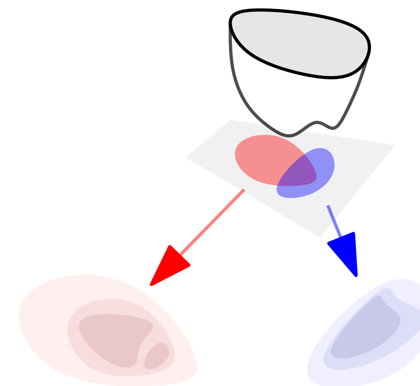


Linear programming



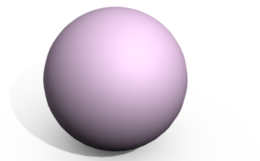
Convex programming



Generalized linear programming

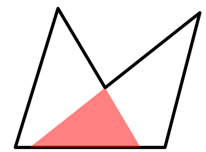
Which families of sets have bounded Helly numbers? What are these bounds?

A whole industry of bounds on Helly numbers (a.k.a "Helly-type theorems").



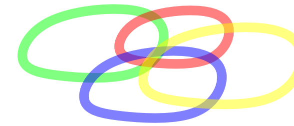
convex sets in $\mathbb{S}^d$

d+2



Star-shapness in the plane

3 [Breen 1985]



Homothets of a convex curve in $\mathbb{R}^2$

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Convexity spaces [Kolodziejczyk 1991], Matroids [Edmonds 2001], ...

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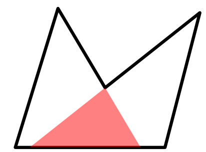

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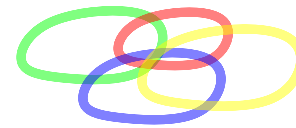

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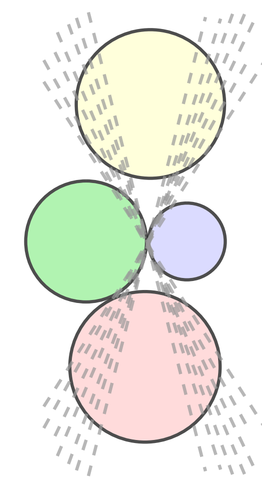
A line transversal to a family is a line that intersects each of its members.

Helly numbers of sets of line transversals to



- disjoint unit disks in $\mathbb{R}^2$: ≤ 5 [Danzer 1957]
- disjoint translates of a convex figure in $\mathbb{R}^2$: ≤ 5 [Tverberg 1989]
- disjoint translates of a convex polyhedron in $\mathbb{R}^3$: unbounded [Holmsen-Matoušek 2004]
- disjoint unit balls in $\mathbb{R}^d$: $\leq 4d-1$ [Cheong-Holmsen-G-Petitjean 2006]

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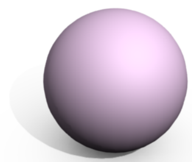

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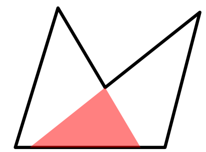

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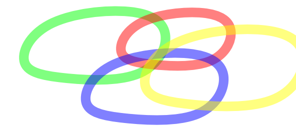

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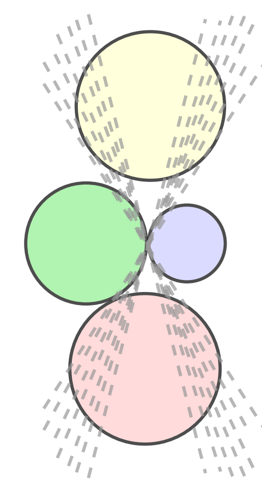

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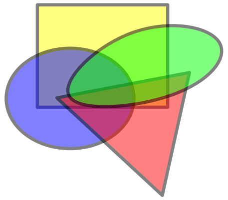
Proofs are technical and somewhat ad hoc.

What systematic conditions could explain these bounds?



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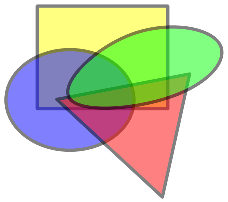

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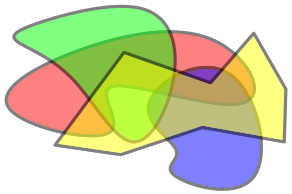

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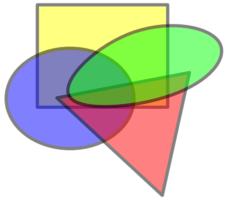

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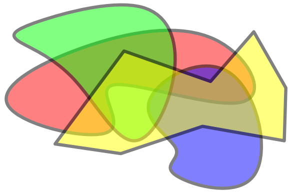

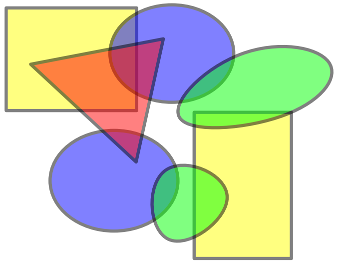

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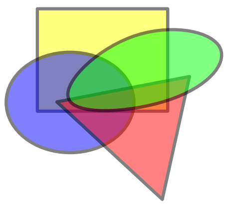

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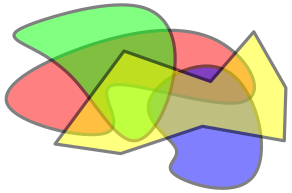

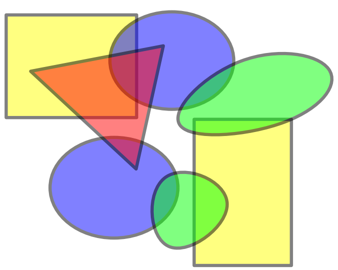

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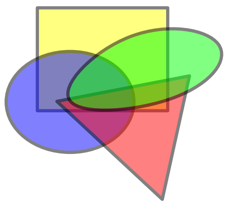

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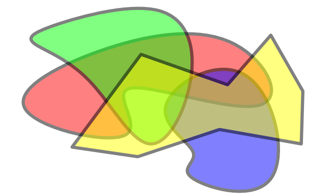

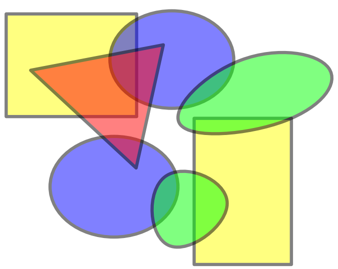

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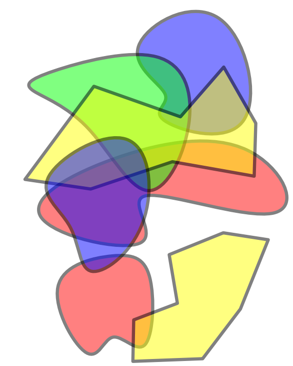

Subsets of $\mathbb{R}^d$ whose intersections have $\leq r$ connected components, each $(\lceil d/2 \rceil - 1)$-connected.
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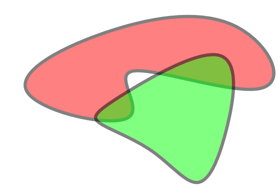

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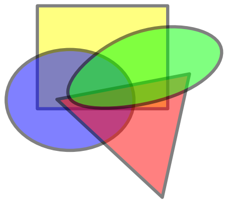

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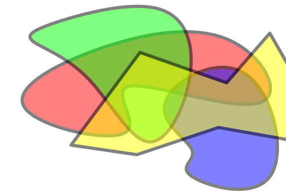

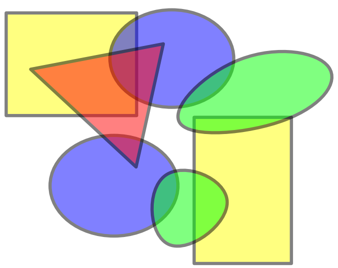

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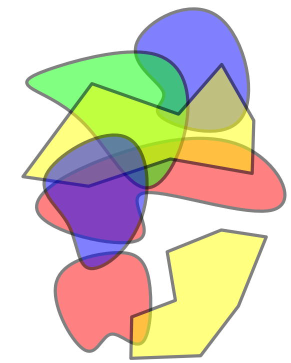

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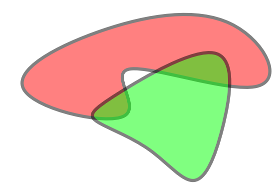

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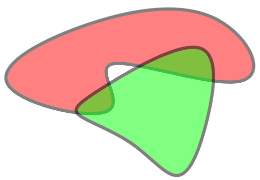
New insights (1/2)

In "reasonable" topological spaces:

$\cap$ of any subfamily has
$\leq r$ connected components,
each homologically trivial

$\Rightarrow$ Helly $\leq r * (\text{max. dim. of a hole in the space}+2)$



[Colin de Verdière-Ginot-G 2014]

Builds on the techniques of [Kalai-Meshulam 2008]

Common derivation of transversal theorems of [Santaló 1940], [Tverberg 1989] and [Cheong+ 2008]

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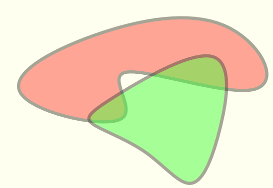
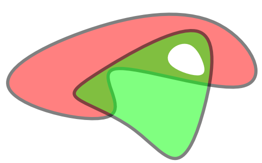


(tomorrow)

New insights (2/2)

In “reasonable” d-dimensional manifolds:

$\cap$ of any subfamily has reduced $\mathbb{Z}_2$-Betti numbers $\leq r$ in dimension $\leq \lceil d/2 \rceil - 1$ $\Rightarrow$ Helly $\leq$ some function of r and d



[G-Paták-Safernová-Tancer-Wagner 2014]
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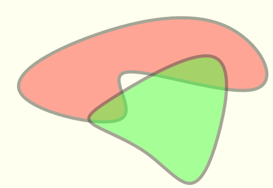


1. Helly numbers are dimensions of holes in nerve (simplicial) complexes

2. Holes in nerve complexes correspond to holes in the union

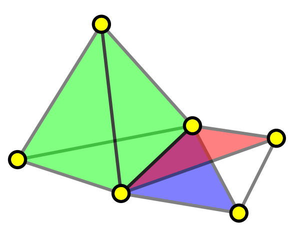
3. Projections with small fibers are well-behaved

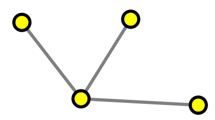
$$\mathrm{Helly}(\mathcal{F}) \leq \left(\begin{array}{c}\text{max. number of}\\ \text{connected components}\\ \text{of } \cap\mathcal{G} \text{ for } \mathcal{G} \subseteq \mathcal{F}\end{array}\right) * \left(\begin{array}{c}\text{max. dimension of}\\ \text{a hole in the space}\end{array} + 2\right)$$

What are simplicial complexes?

geometric simplicial complex

"A collection of geometric simplices in $\mathbb{R}^d$ such that any two are disjoint or intersect in a common face."





abstract simplicial complex

"A collection of sets that is closed under taking subsets."

$$\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}\}$$

What are simplicial complexes?

set of vertices forming a geometric simplex

geometric simplicial complex

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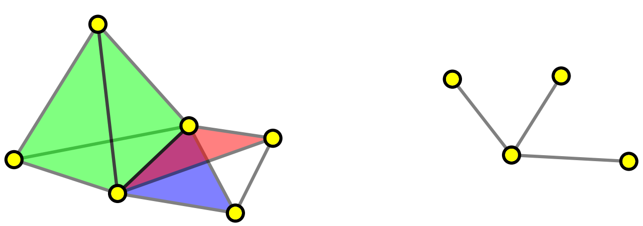

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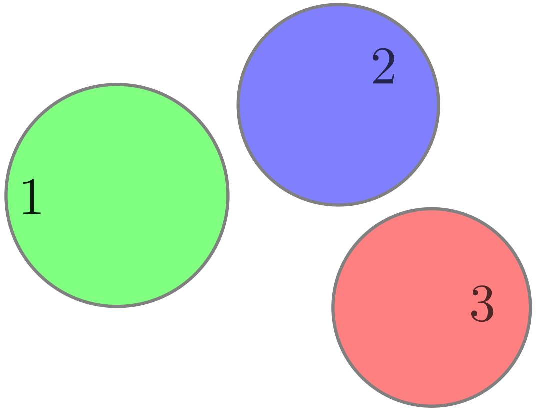
geometric realization

map singletons to points in general position in $\mathbb{R}^d$, d large enough

take convex hulls of points corresponding to abstract simplices

The nerve $\mathcal{N}(\mathcal{F})$ of a family $\mathcal{F}$ of sets is $\mathcal{N}(\mathcal{F}) = \{\mathcal{G} : \mathcal{G} \subseteq \mathcal{F} \text{ and } \cap \mathcal{G} \neq \emptyset\}$.

$\mathcal{F}$



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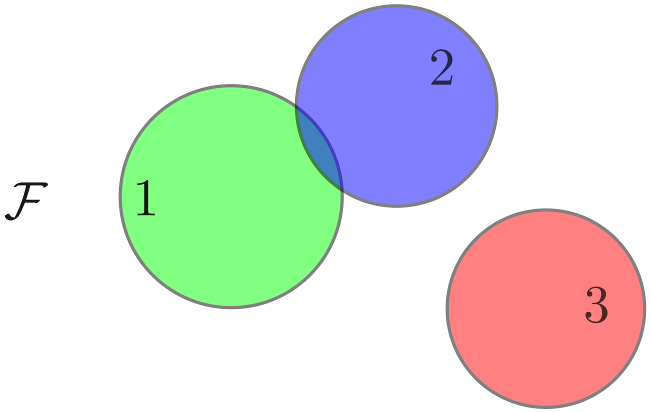

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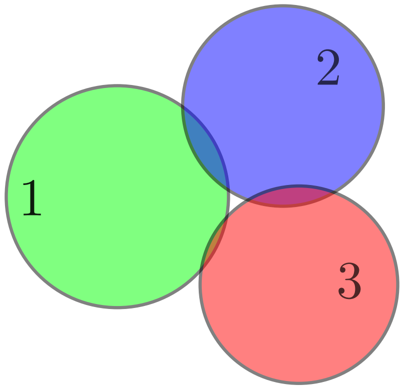

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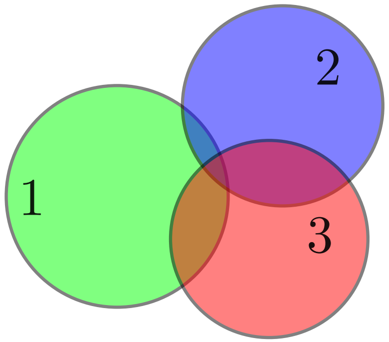

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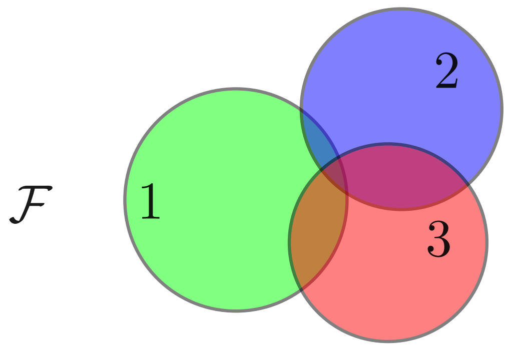

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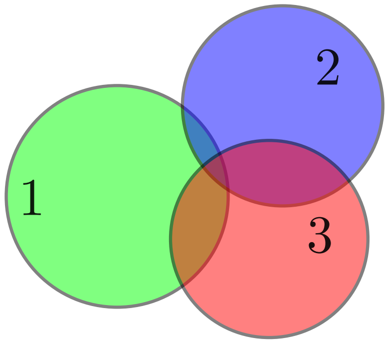

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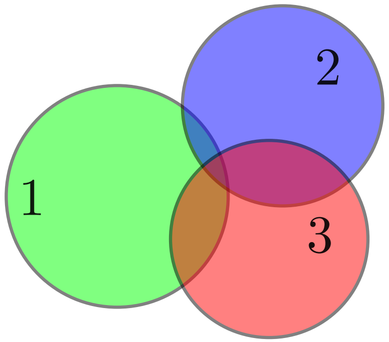

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boundary of a $(|\mathcal{G}| - 1)$-simplex

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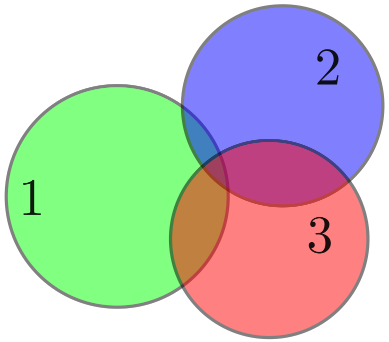

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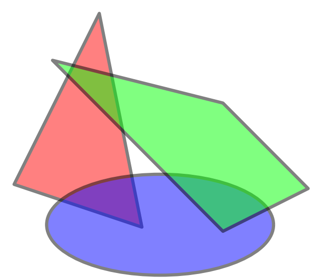
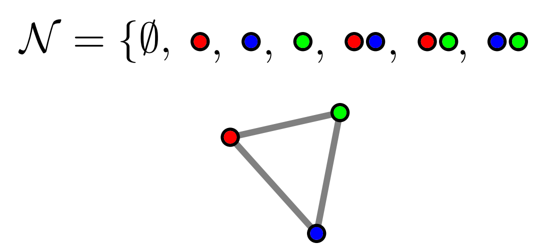
$\mathcal{F}$ has Helly number $\geq h$
$\Leftrightarrow$
$\mathcal{N}(\mathcal{F})$ contains the boundary of a k-simplex for $k \geq h - 1$





Families with large Helly number have nerves with "holes" of large dimension.

What is a hole?

homotopy theory expresses algebraically how continuous images of k-spheres extends into continuous images of k-balls.

homology theory expresses which submanifolds are not boundaries of submanifolds.

Do not capture exactly the same notions.

Nuances not essential for many applications to discrete geometry.

The Leray number of a simplicial complex K with vertex set V is

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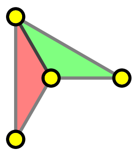
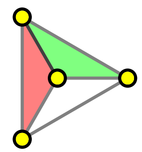
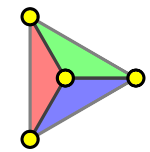
L ()

L ()

L ()

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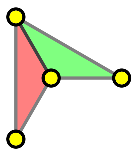
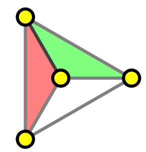
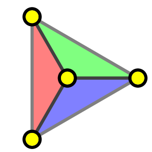
L $\Bigg($  $\Bigg) = 1$

L $\Bigg($  $\Bigg)$

L $\Bigg($  $\Bigg)$

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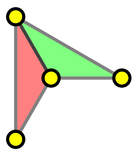
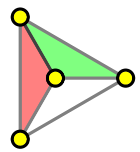
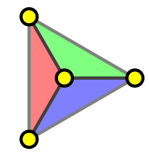
L () $= 1$

L () $= 2$

L ()

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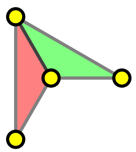
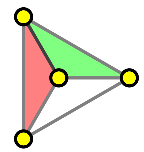
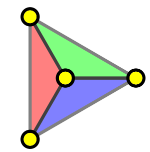
L $\Bigg($  $\Bigg) = 1$

L $\Bigg($  $\Bigg) = 2$

L $\Bigg($  $\Bigg) = 2$

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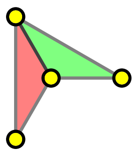
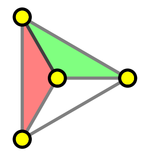
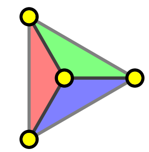
L () $= 1$

L () $= 2$

L () $= 2$

Lemma. For any family $\mathcal{F}$ of sets, $\text{Helly}(\mathcal{F}) \leq L(\mathcal{N}(\mathcal{F})) + 1$

Proof: Pick $\mathcal{G} \subseteq \mathcal{F}$ of maximum size such that $\cap\mathcal{G} = \emptyset$, and $\forall A \in \mathcal{G}, \cap(\mathcal{G} \setminus A) \neq \emptyset$.

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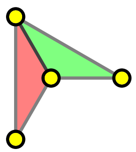
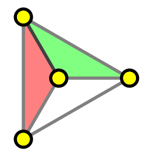
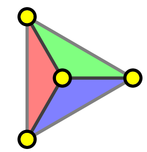
L () $= 1$ L () $= 2$ L () $= 2$

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Bounding $L(\mathcal{N}(\mathcal{F}))$ also gives a fractional Helly theorem, an ε-net theorem, a (p, q)-theorem for the intersection-closure of $\mathcal{F}$.

[Alon-Kalai-Matoušek-Meshulam 2002]

1. Helly numbers are dimensions of holes in nerve (simplicial) complexes

2. Holes in nerve complexes correspond to holes in the union

3. Projections with small fibers are well-behaved

$$\mathrm{Helly}(\mathcal{F}) \leq \begin{pmatrix} \text{max. number of} \\ \text{connected components} \\ \text{of } \cap\mathcal{G} \text{ for } \mathcal{G} \subseteq \mathcal{F} \end{pmatrix} * \left(\begin{matrix} \text{max. dimension of} \\ \text{a hole in the space} \end{matrix} + 2 \right)$$

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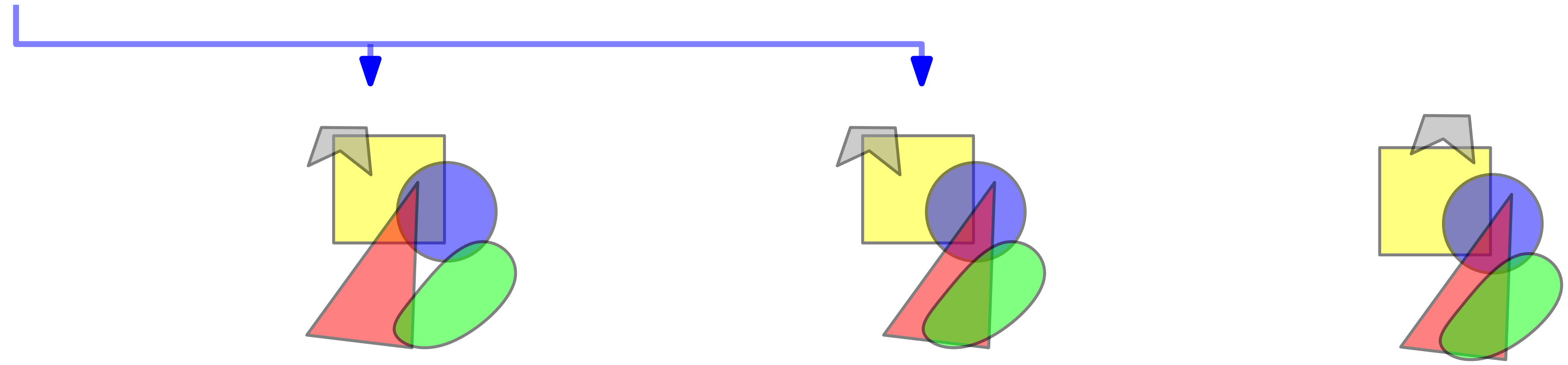

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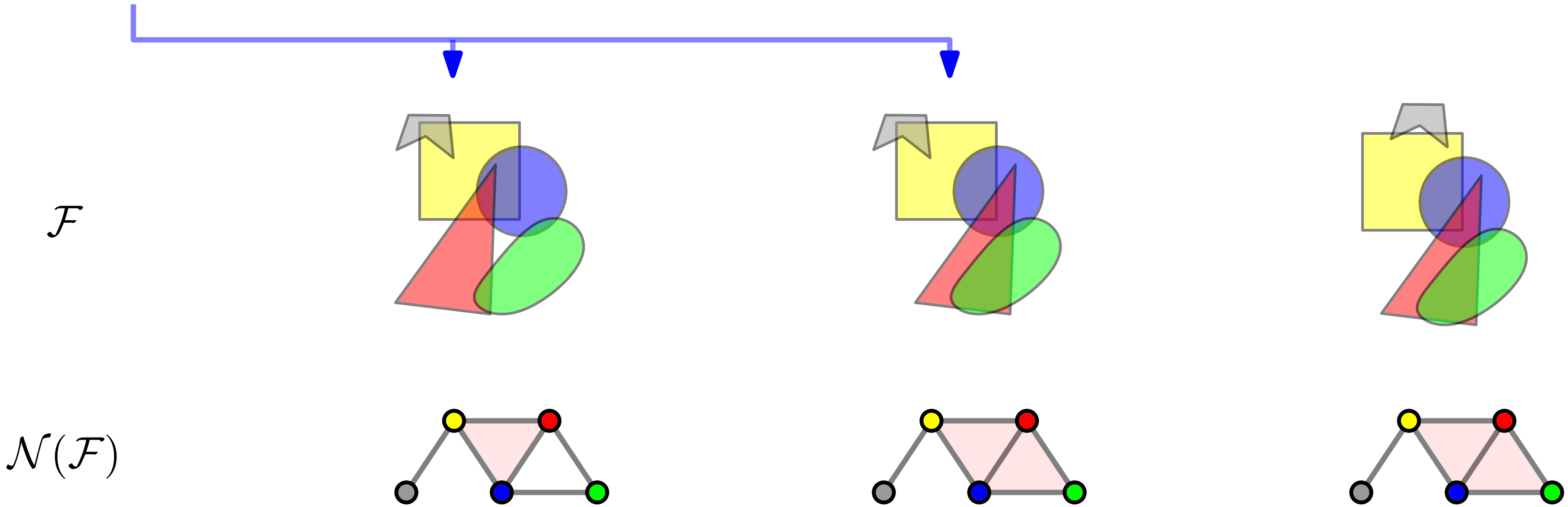

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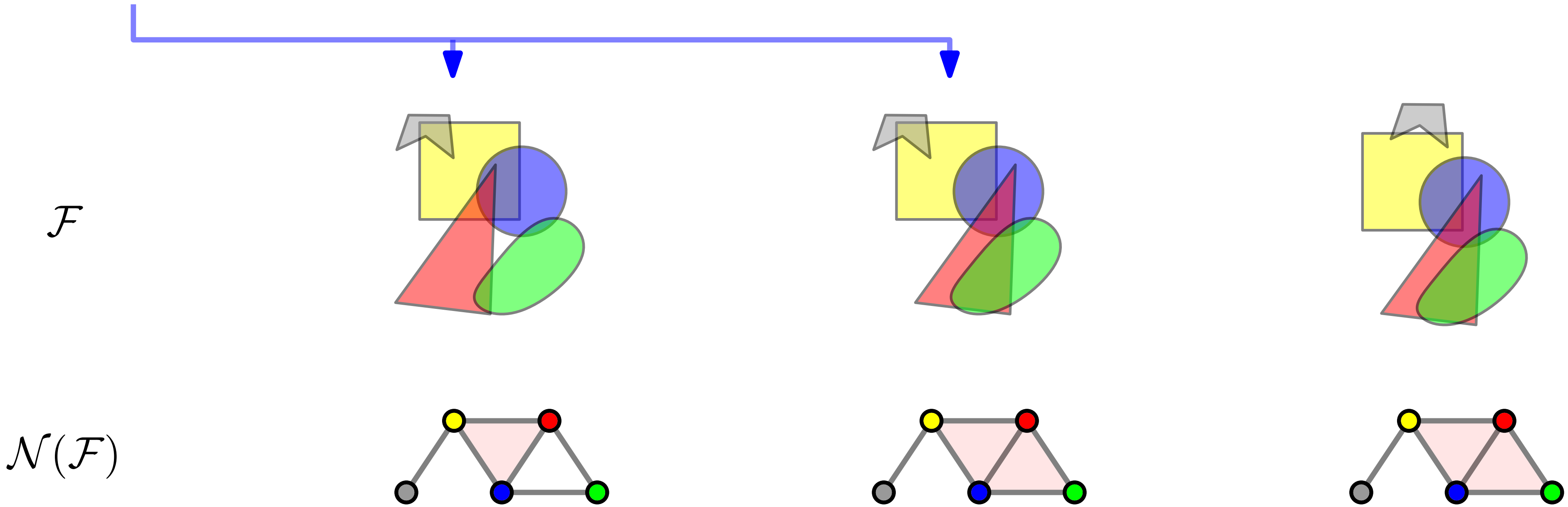

Nerve Theorem. [Borsuk 1948, Leray 1945] If $\mathcal{F}$ is a good cover in a triangulable space then $|\mathcal{N}(\mathcal{F})|$, the geometric realization of $\mathcal{N}(\mathcal{F})$, is homotopy-equivalent to $\cup\mathcal{F}$.

Holes in the nerve $\rightsquigarrow$ hole in a subset of the ambient space

Can a subset of $\mathbb{R}^d$ have holes of dimension more than $d-1$?

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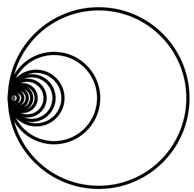

If $k \geq 2$ then $H_i(\circledcirc_k, \mathbb{Q})$ is nontrivial for all $i \equiv 1 \mod k-1$ [Barratt-Milnor 1962]
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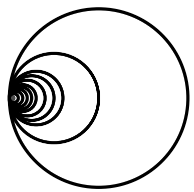

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Lemma Any open subset of a (paracompact) manifold of dimension d has trivial $\mathbb{Q}$-homology in any dimension $i \geq d+1$. If the manifold is non-compact or non-orientable then this bound improves to d.

So Helly numbers $\rightsquigarrow$ holes in nerves $\rightsquigarrow$ holes in union looks promising

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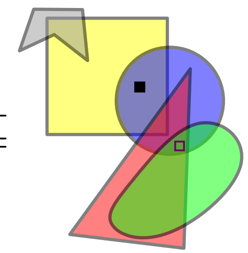
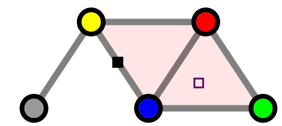
Proof sketch: Build the blow-up complex $C \subseteq$  $\times$ 

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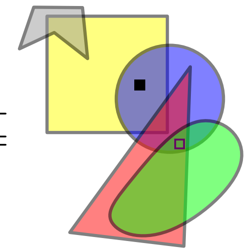
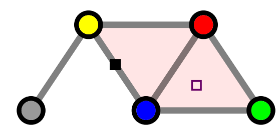
Proof sketch: Build the blow-up complex $C \subseteq$  $\times$ 

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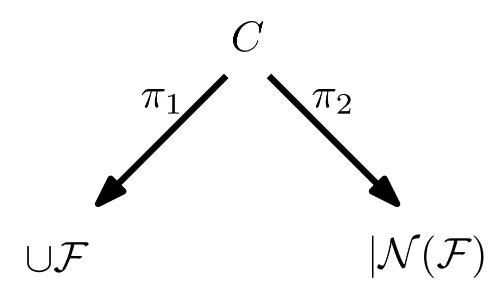
These projections have contractible fibers



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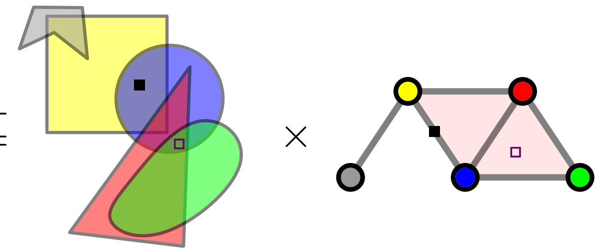

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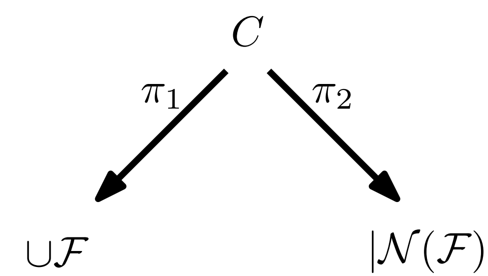
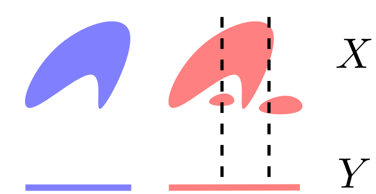


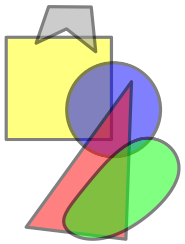
The Vietoris-Begle mapping theorem yields that $C \simeq \cup\mathcal{F}$ and $C \simeq |\mathcal{N}(\mathcal{F})|$ $\square$

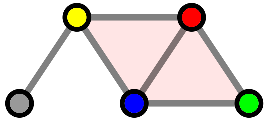
The "**Vietoris-Begle mapping theorem**" asserts that if X, Y are "nice" topological spaces and $\pi : X \to Y$ is continuous, surjective, with contractible fibers and nice then $X \simeq Y$.



$\mathcal{F}$ is acyclic if the intersection of any subfamily is empty or a disjoint union of $\mathbb{Q}$-homology cells.

Like good cover in homology, but allowing multiple connected components





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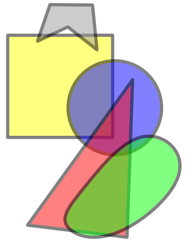
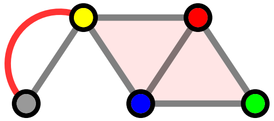
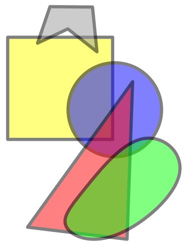


Fix the nerve by adding multiple simplices in case of multiple components





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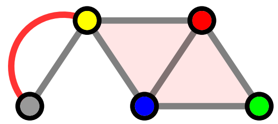

The multinerve of $\mathcal{F}$, denoted $\mathcal{M}(\mathcal{F})$, is the poset

$$\mathcal{M}(\mathcal{F}) = \{(\mathcal{G}, X) \mid \mathcal{G} \subseteq \mathcal{F}, X \text{ is a connected component of } \cap \mathcal{G}\}$$

$$\text{ordered by } (\mathcal{G}, X) \prec (\mathcal{G}', X') \text{ iff } \mathcal{G} \subset \mathcal{G}' \text{ and } X \supset X'.$$

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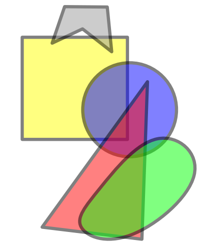

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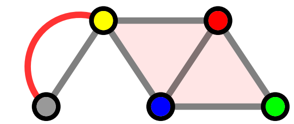

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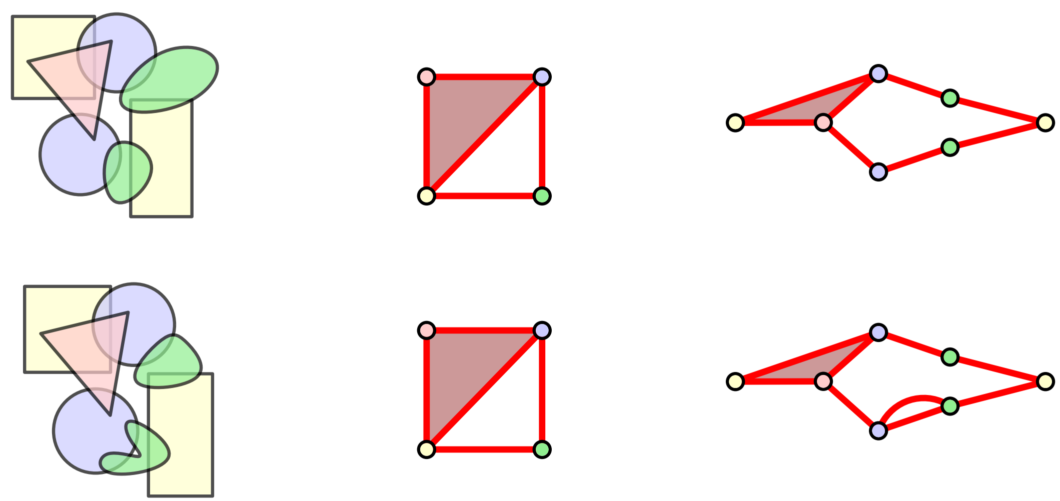

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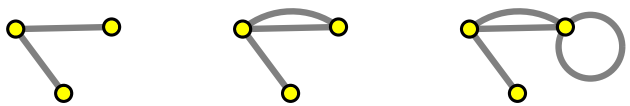
Simplicial poset: every lower-interval is isomorphic to a the face lattice of a simplex.



Can define (topological) geometric realization, simplicial homology...

Simplices no longer defined by their vertices; explicit incidences...

Arguments about co-faces may not extend

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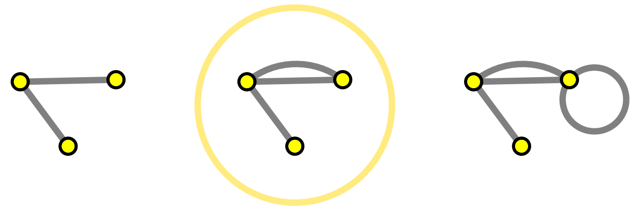

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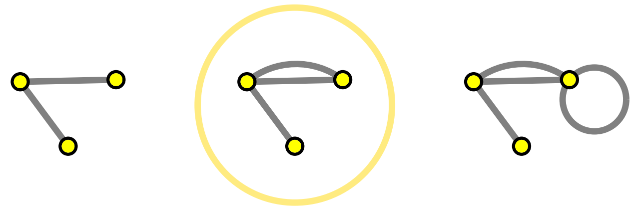

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Theorem 1. If $\mathcal{F}$ is an acyclic family of open sets in a locally arc-wise connected topological space then $\forall i \geq 0, \quad \tilde{H}_i(\mathcal{M}(\mathcal{F}), \mathbb{Q}) \cong \tilde{H}_i(\cup\mathcal{F}, \mathbb{Q})$.

Proof: A blow-up complex / Vietoris-Begle mapping theorem approach works (even in homotopy). □

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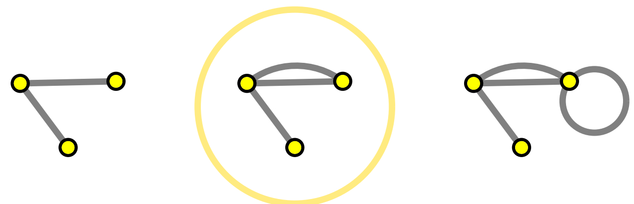

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Proof (bis): Interpret the multinerve as a Čech chain complex and use Leray's acyclic cover theorem.
Generalized Mayer-Vietoris principle, spectral sequences... □

Theorem 2. Let $\mathcal{F}$ be a family of open sets in a locally arc-wise connected topological space. Let $s \in \mathbb{N}$ and assume $\tilde{H}_i(\cap\mathcal{G}, \mathbb{Q}) = 0$ for any $\mathcal{G} \subseteq \mathcal{F}$ and any $i \geq \max(1, s - |\mathcal{G}|)$. Then $\tilde{H}_i(\mathcal{M}(\mathcal{F}), \mathbb{Q}) \cong \tilde{H}_i(\cup\mathcal{F}, \mathbb{Q})$ for $\ell = 0$ and any $\ell \geq s$.

If we care only about high-dimensional homology, we can allow non-trivial low-dimensional homology in intersections of few objects
[Hell 2005 and 2006]

1. Helly numbers are dimensions of holes in nerve (simplicial) complexes

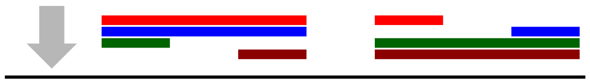
2. Holes in nerve complexes correspond to holes in the union

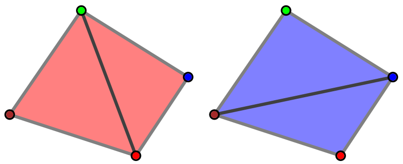
3. Projections with small fibers are well-behaved

$$\mathrm{Helly}(\mathcal{F}) \leq \begin{pmatrix} \text{max. number of} \\ \text{connected components} \\ \text{of } \cap\mathcal{G} \text{ for } \mathcal{G} \subseteq \mathcal{F} \end{pmatrix} * \left(\begin{matrix} \text{max. dimension of} \\ \text{a hole in the space} \end{matrix} + 2 \right)$$

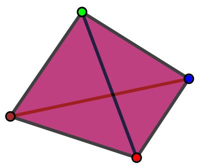
multinerve theorem $\Rightarrow \quad L(\mathcal{M}(\mathcal{F})) \leq \left(\begin{array}{c}\text{max. dimension of}\\ \text{a hole in the space}\end{array} +1\right)$

... but $L(\mathcal{M}(\mathcal{F}))$ does not bound $\mathrm{Helly}(\mathcal{F})$



$\mathcal{M} =$ 

$\simeq \quad \bullet\bullet$

$\mathcal{N} =$ 

$\simeq \mathbb{S}^2$

$\mathcal{N}(\mathcal{F}) = \{\mathcal{G} : \mathcal{G} \subseteq \mathcal{F}, \cap\mathcal{G} \neq \emptyset\}$

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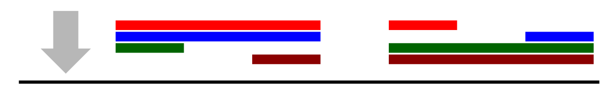

Can we understand how

$$\pi : \begin{cases} \mathcal{M}(\mathcal{F}) & \rightarrow & \mathcal{N}(\mathcal{F}) \\ (\mathcal{G}, X) & \mapsto & \mathcal{G} \end{cases}$$

"transports" the Leray number (or similar quantities)?

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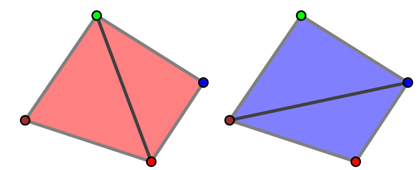

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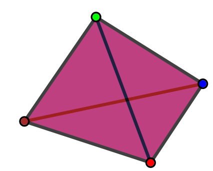

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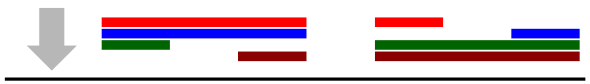

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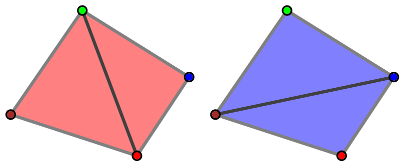

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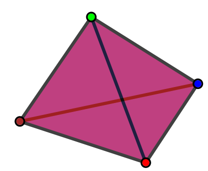

$\simeq \mathbb{S}^2$

π is well-behaved:

- simplicial, surjective
- maps a k-simplex to a k-simplex
- is at most r-to-one where $r = \max_{\mathcal{G} \subseteq \mathcal{F}} \#cc(\cap \mathcal{G})$

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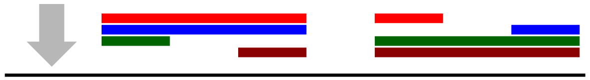
Such maps can be found in a broader setting:

Theorem. [Eckhoff-Nischke 2009] Let $\mathcal{G}$ be non-additive and intersection-closed. If every intersection of members of $\mathcal{F}$ is a disjoint union of at most r members of $\mathcal{G}$ then $\mathrm{Helly}(\mathcal{F}) \leq r\mathrm{Helly}(\mathcal{G})$.

Conjectured by [Grünbaum-Motzkin 63]



$\mathcal{G}$ = Convex sets in $\mathbb{R}^d$

$r(d+1)$, [Amenta 1996]

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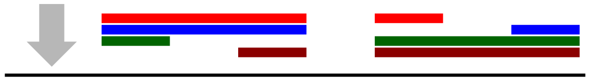

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Intersection-closed and non-additive $\Rightarrow$ components over $\mathcal{G}$ are well-defined.

Let $D_1, D_2 \subseteq \mathcal{G}$ with $\cup D_1 = \cup D_2$. Pick $A \in D_1$ and write $A = \cup_{B \in D_2} A \cap B$.

The $A \cap B$'s are in $\mathcal{G}$ $\Rightarrow$ at most one $A \cap B$ is nonempty

Symmetric argument with $B \Rightarrow B = A$

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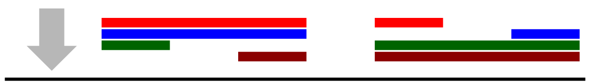

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Symmetric argument with $B \Rightarrow B = A$

There is an underlying nice projection from $\mathcal{N}(\mathcal{G})$ to $\mathcal{N}(\mathcal{F})$

Map every element of $\mathcal{G}$ to the element of $\mathcal{F}$ it is a component over $\mathcal{G}$ of
This map extends into a simplicial map $\pi : \mathcal{N}(\mathcal{G}) \to \mathcal{N}(\mathcal{F})$
π is dimension-preserving and at most r-to-one

Let X and Y be simplicial complexes.
Let $\pi : X \to Y$ be a surjective, dimension preserving, $\leq r$-to-one simplicial map.

dimension-preserving: the image of a simplex is a simplex of the same dimension

at most r-to-one: the fiber of every simplex of Y has cardinality at most r

Theorem. [Kalai-Meshulam 2008] $L(Y) + 1 \leq r(L(X) + 1)$.

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Define $H(K)$ as the maximum dimension of a simplicial hole of K.

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A "good" filtration of K is a sequence $\emptyset = K_0 \subset K_1 \subset \ldots \subset K_m = K$ such that

(i) each K_i is a simplicial complex

(ii) each $K_i \setminus K_{i-1}$ has a unique inclusion-maximal element

Define $\Delta(K)$ as the maximum dimension, over all "good" filtrations of K, of a simplicial hole in K_i for some $i < m$ that is not a simplicial hole in K.

Theorem. [Amenta 1996] $\Delta(Y) + 1 \leq r\Delta(X)$.

Let X be a simplicial poset and Y a simplicial complex.
Let $\pi : X \to Y$ be a surjective, dimension preserving, $\leq r$-to-one simplicial map.

Question. is $L(Y) + 1 \leq r(L(X) + 1)$?

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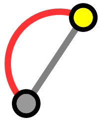
For a simplicial complex K, $\forall i \geq L(K)$ and $\forall \sigma \in K$, $\tilde{H}_i(\mathit{lk}_K(\sigma), \mathbb{Q}) = 0$

Links behave differently for simplicial complexes and posets.



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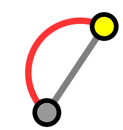

Define the J-index of a simplicial poset K with vertex set V as

$$J(K) = \min\{\ell \in \mathbb{N} : \quad \forall i \geq \ell, \forall S \subseteq V, \forall \sigma \in K, \quad \tilde{H}_i(\dot{D}_{K[S]}(\sigma), \mathbb{Q}) = 0\}$$

$\dot{D}_K(\sigma)$ is the order complex of $[\sigma, \cdot)$, a sub-complex of sd$K$

$J(X) = L(X)$ if X is a simplicial complex. [Kalai-Meshulam 2006]

The multinerve theorem bounds J of multinerves of acyclic families.

Theorem 3. $L(Y) + 1 \leq r(J(X) + 1)$.

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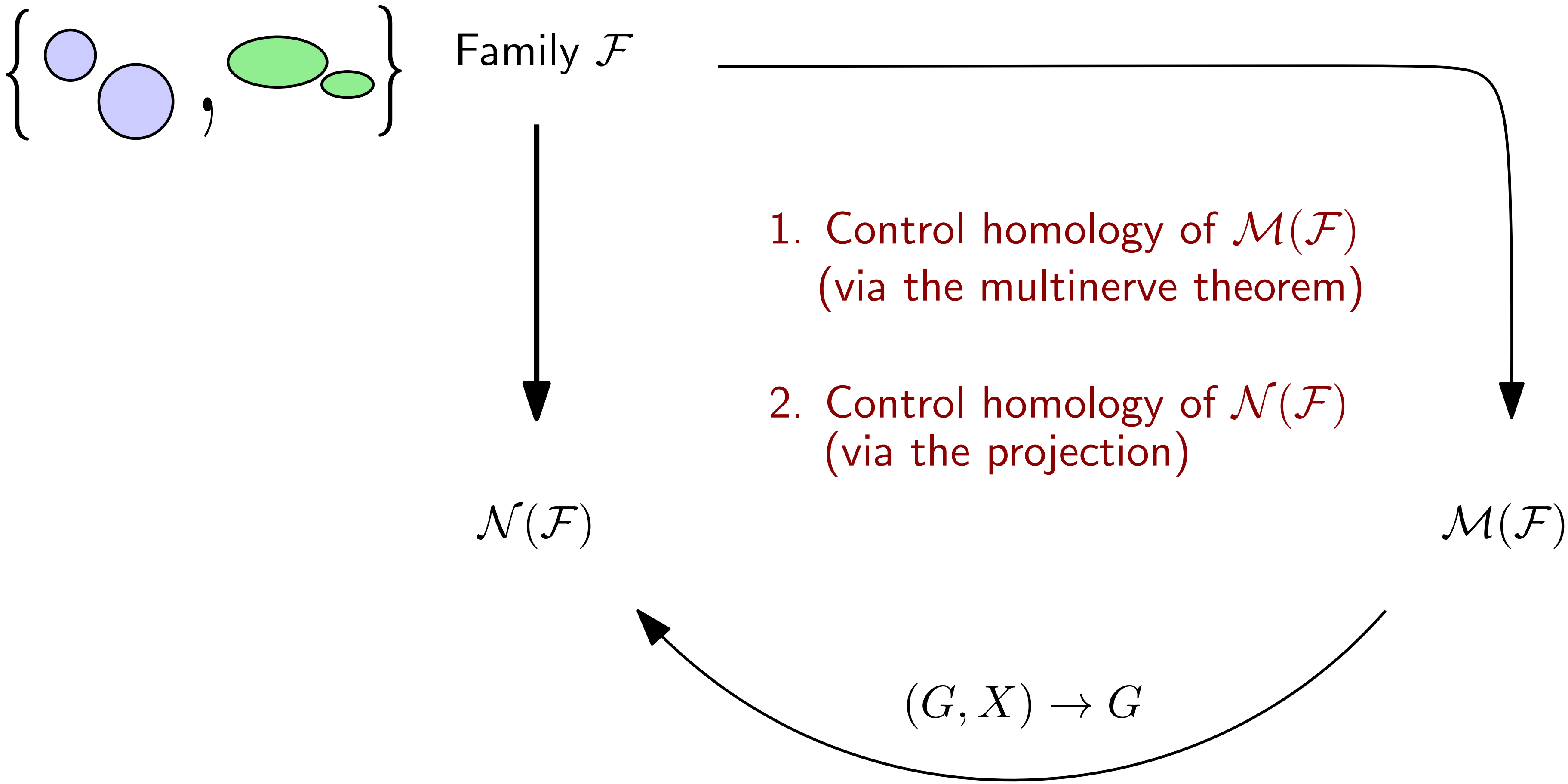

Theorem 4. If $\mathcal{F}$ is a finite family of open subsets of a locally arc-wise connected topological space Γ such for every subfamily $\mathcal{G}$ of size at least t the intersection $\cap\mathcal{G}$ has at most r connected components, each with trivial homology in dimension $\max(1, s - |\mathcal{G}|)$ and more, then $\mathrm{Helly}(\mathcal{F}) \leq r(\max(d_\Gamma, s, t) + 1)$.

d_Γ is the dimension of vanishing $\mathbb{Q}$-homology for open sets in Γ

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Γ a non-compact sub-manifold of $\mathbb{G}_{2,d+1}$, the Grassmannian of lines in $\mathbb{R}^d$ so $d_\Gamma = \dim(\Gamma) = 2d - 2$.

Line transversals to ≥ 2 convex planar figures or balls are acyclic.

Numbers of cc of line transversals to convex planar figures or balls are counted by geometric permutations.

$s = d + 1$ to account for transversals to a convex $\simeq \mathbb{RP}^{d-1}$.

t used to optimize the use of bounds on number of geometric permutations.

Shape	Previous bound	Our bound	d_Γ	s	t	r
Parallelotopes in $\mathbb{R}^d$ $(d \geq 2)$	$2^{d-1}(2d-1)$ [Santaló 1940]	$2^{d-1}(2d-1)$	$2d-2$	$d+1$	1	2^{d-1}
Disjoint translates of a planar convex figure	5 [Tverberg 1989]	10	2	3	4	2
Disjoint unit balls in $\mathbb{R}^d$:						
$d = 2$	5 [Danzer 1957]	12	$2d-2$	$d+1$	1	3
$d = 3$	11 [Cheong+ 2008]	15	$2d-2$	$d+1$	1	3
$d = 4$	15 [Cheong+ 2008]	20	$2d-2$	$d+1$	9	2
$d = 5$	19 [Cheong+ 2008]	20	$2d-2$	$d+1$	9	2
$d \geq 6$	$4d-1$ [Cheong+ 2008]	$4d-2$	$2d-2$	$d+1$	9	2

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$d \geq 6$	$4d-1$ [Cheong+ 2008]	$4d-2$	$2d-2$	$d+1$	**7**	2

To summarize...

The Helly number of a family $\mathcal{F}$ of sets is the maximum size of a minimum sub-family of $\mathcal{F}$ with empty intersection.

$$\text{Helly}(\mathcal{F}) = \max\{|\mathcal{G}| : \mathcal{G} \subseteq \mathcal{F}, \cap\mathcal{G} = \emptyset, \forall A \in \mathcal{G}, \cap(\mathcal{G} \setminus A) \neq \emptyset\}$$

1. Helly numbers are dimensions of holes in nerve complexes

$$\mathcal{N}(\mathcal{F}) = \{\mathcal{G} : \mathcal{G} \subseteq \mathcal{F} \text{ and } \cap\mathcal{G} \neq \emptyset\}$$

$\mathcal{F}$ has Helly number $\geq h \Leftrightarrow \mathcal{N}(\mathcal{F})$ contains the boundary of a k-simplex for $k \geq h-2$

Use Leray numbers: $L(K) = \min\{\ell \in \mathbb{N} : \quad \forall i \geq \ell, \forall S \subseteq V, \quad \tilde{H}_i(K[S], \mathbb{Q}) = 0\}$

2. Holes in nerve complexes correspond to holes in the union

Nerve theorem for good covers

Multinerve and multinerve theorem for acyclic families

Vietoris-Begle mapping theorem

3. Projections with small fibers are well-behaved

Underlying the (partial) proofs of the Grünbaum-Motzkin conjecture

Somewhat extend to maps between simplicial posets

$$\text{Helly}(\mathcal{F}) \leq \left(\begin{array}{c}\text{max. number of}\\ \text{connected components}\\ \text{of } \cap\mathcal{G} \text{ for } \mathcal{G} \subseteq \mathcal{F}\end{array}\right) * \left(\begin{array}{c}\text{max. dimension of}\\ \text{a hole in the space}\end{array} + 2\right)$$

Common derivation of transversal theorems of [Santaló 1940], [Tverberg 1989] and [Cheong+ 2008]

End of part I

Helly numbers and topological complexity

Part II

The Helly number of a family $\mathcal{F}$ of sets is the maximum size of a minimum sub-family of $\mathcal{F}$ with empty intersection.

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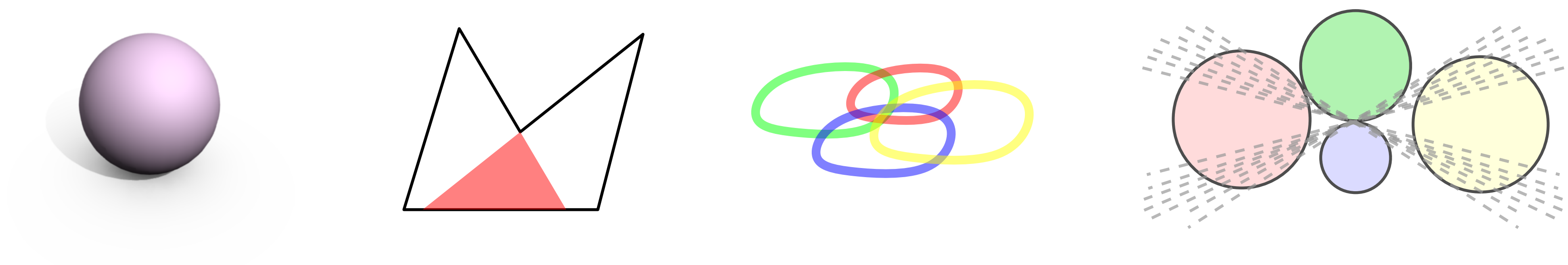
Many Helly-type theorems.



Goal: some common topological explanation.

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2. Holes in nerve complexes correspond to holes in the union

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Vietoris-Begle mapping theorem

3. Dimension-preserving, bounded degree projections are well-behaved

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4. Bounds on Helly numbers arise from non-embeddability

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$\tilde{\beta}_i(\cap\mathcal{G}) \leq b$ for all $\mathcal{G} \subseteq \mathcal{F}$ and $i \leq \lceil d/2 \rceil - 1$ $\Rightarrow$ $\mathrm{Helly}(\mathcal{F})$ is bounded by some function of d and b

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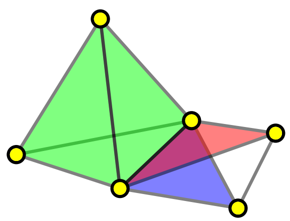
Let K be a simplicial complex with geometric realization $|K|$.

An embedding of K into $\mathbb{R}^d$ is a map from $|K|$ into $\mathbb{R}^d$ that is an homeomorphism on its image.

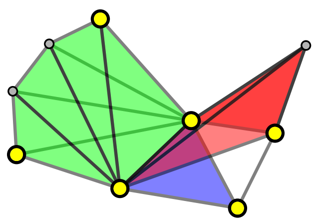
Map singletons to points, edges to arcs, triangles to disks... satisfying boundary conditions.

Images of simplices intersect in exactly the image of their common face

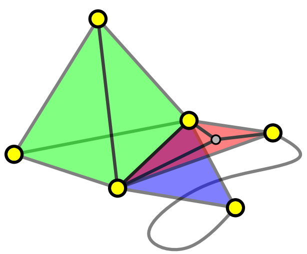
Linear embeddings



Piece-wise linear embeddings



Topological embeddings



$\Delta_m^{(t)} = \binom{[m+1]}{t+1}$ *is the t-dimensional skeleton of the m-dimensional simplex*

$[x] = \{1, 2, \ldots, x\}$ *and* $\binom{[x]}{t}$ = *all t-elements subsets of* $[x]$

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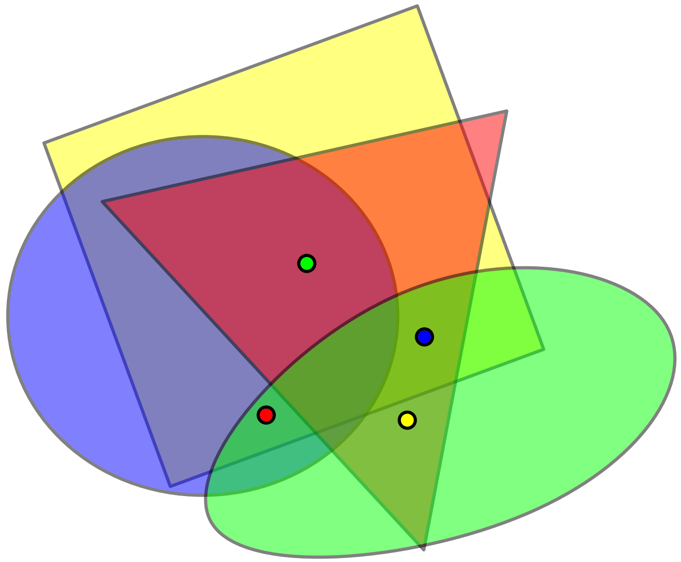

Helly from Radon

Let $\mathcal{F} = \{A_1, A_2, \ldots A_k\}$ be convex sets in $\mathbb{R}^d$ such that $k \geq d+2$ and $\forall j \leq k$, $\cap_{i \neq j} A_i \neq \emptyset$

Pick $p_j \in \cap_{i \neq j} A_i$

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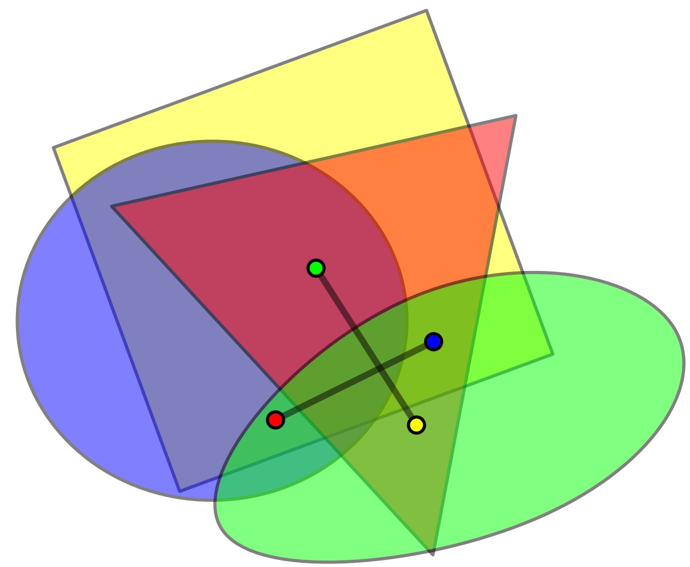

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Pick $p_j \in \cap_{i \neq j} A_i$

There exists a partition $X \cup Y$ of $\{p_1, p_2, \ldots, p_k\}$ and $h \in \mathrm{conv}(X) \cap \mathrm{conv}(Y)$

$h \in (\cap_{i:p_i \notin X} A_i) \cap (\cap_{i:p_i \notin Y} A_i) = \cap\mathcal{F}$

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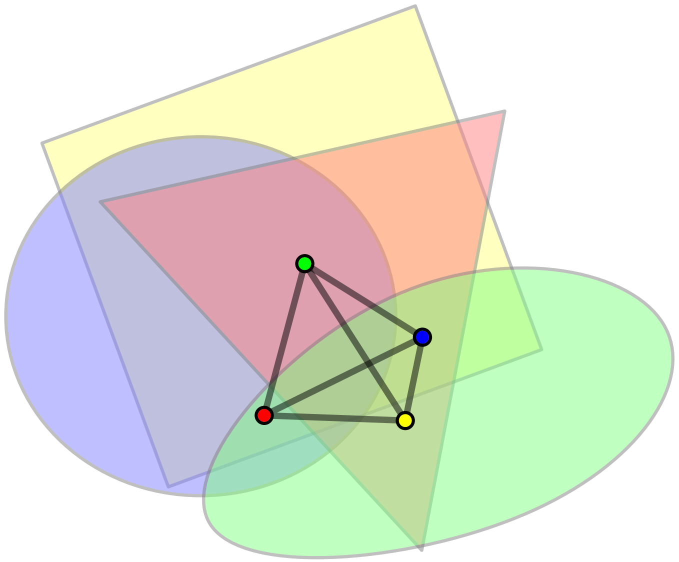

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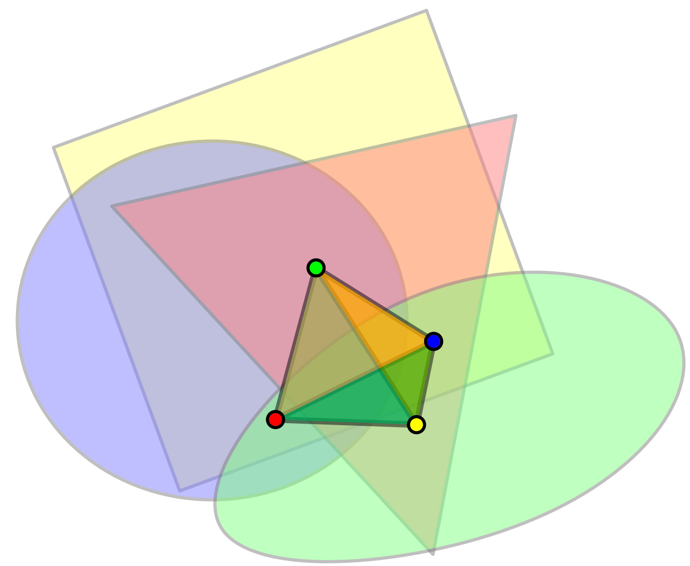

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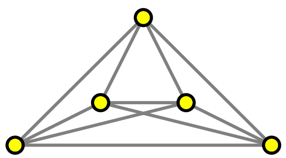
Non-planarity of $K_5 \Rightarrow$ Helly number for path-connected intersections in $\mathbb{R}^2$.

$\Delta_n^{(1)} \not\hookrightarrow \mathbb{R}^2$ for $n \geq 5$



Corollary. If $\mathcal{F}$ is a family of sets in $\mathbb{R}^2$ such that the intersection of any subfamily is empty or path-connected then $\mathrm{Helly}(\mathcal{F}) \leq 4$.

Proof: Let $\mathcal{F} = \{A_1, A_2, \dots A_k\}$ such that $k \geq 5$ and $\forall j \leq k$, $\cap_{i \neq j} A_i \neq \emptyset$

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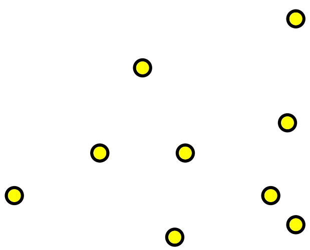

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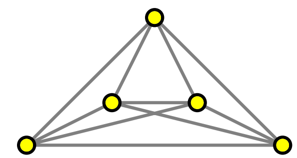

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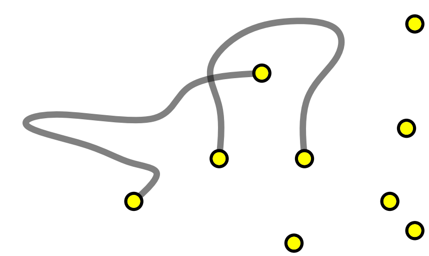
Connect every p_a and p_b inside $\cap_{i \neq a,b} A_i$



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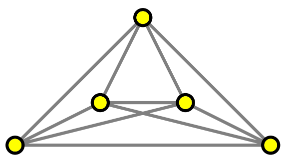

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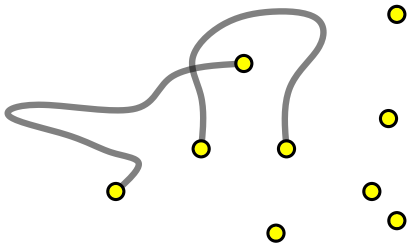
Connect every p_a and p_b inside $\cap_{i \neq a,b} A_i$

Two edges $p_a p_b$ and $p_u p_v$ cross, with $\{a,b\} \cap \{u,v\} = \emptyset$.

The intersection point belongs to $(\cap_{i \neq a,b} A_i) \cap (\cap_{i \neq u,v} A_i) = \cap_i A_i$. $\square$



$\not\hookrightarrow$	Assumption on nonempty intersections	Bound on the Helly number
Topological Radon: $\Delta_{d+1}^{(d)} \not\hookrightarrow \mathbb{R}^d$ [Bajmóczy-Bárány 1979]	contractible	$d+1$
$\Delta_{2\lceil d/2\rceil+2}^{(\lceil d/2\rceil)} \not\hookrightarrow \mathbb{R}^d$ [Van Kampen 1931, Flores 1932]	$\lceil d/2\rceil$-connected	$2\lceil d/2\rceil+2$

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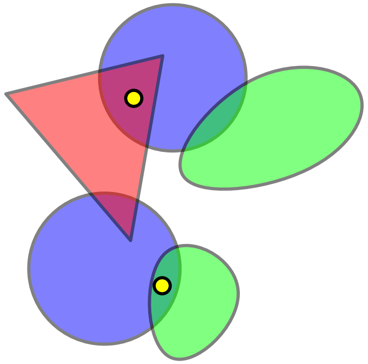
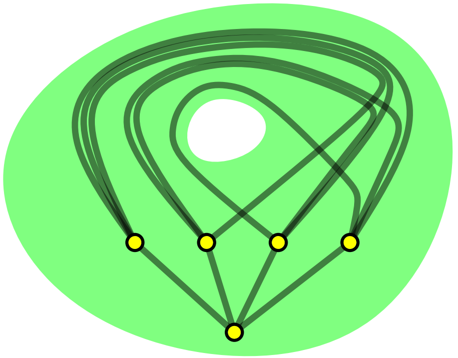
Can we allow some disconnection?





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Ramsey's theorem. For any x, y and z there exists $R = R_{x,y,z} \in \mathbb{N}$ such that any coloring of the complete x-uniform hypergraph on at least R vertices by y colors contains z vertices inducing a monochromatic sub-hypergraph.

complete x-uniform hypergraph: all subsets of size x of a finite set

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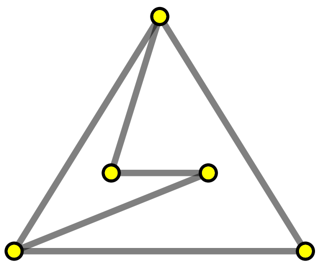
Lemma. Let G be a graph on n vertices where any 3 vertices span at least one edge. If $n \geq R_{3,3,9}$ then G contains K_5 as an induced subgraph.



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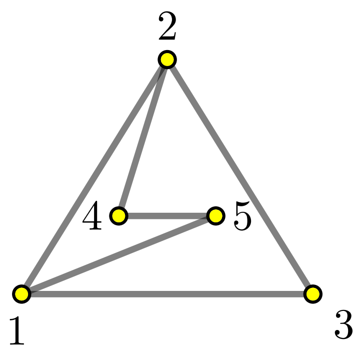

Proof: Number the vertices $1, 2, \ldots, n$.

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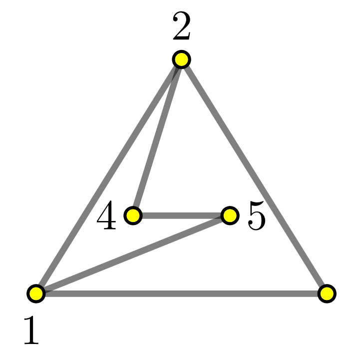

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Ramsey's theorem. For any x, y and z there exists $R = R_{x,y,z} \in \mathbb{N}$ such that any coloring of the complete x-uniform hypergraph on at least R vertices by y colors contains z vertices inducing a monochromatic sub-hypergraph.

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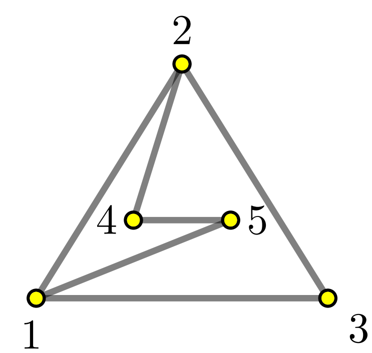
This colors the complete 3-uniform hypergraph by $\{1,2\}, \{1,3\}$ and $\{2,3\}$.

For $n \geq R_{3,3,9}$ some 9 vertices span triples all colored by the same pair $\{a, b\}$.



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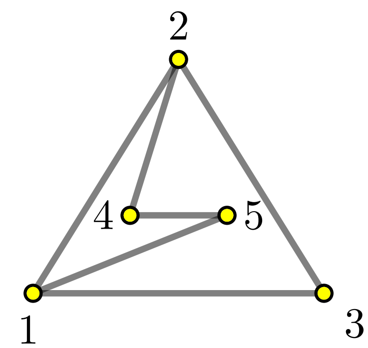

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If $\{a, b\} = \{1, 2\}$ then the vertices with rank $\{1,2,3,4,5\}$ span a K_5.

$\cdots$ $\{2,3\}$ $\cdots$ $\{2,3,4,5,6\}$ $\cdots$

$\cdots$ $\{1,3\}$ $\cdots$ $\{1,3,5,7,9\}$ $\cdots$ $\square$

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The intersection point of these edges lies in $\cap \mathcal{F}$. $\square$

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We actually proved:

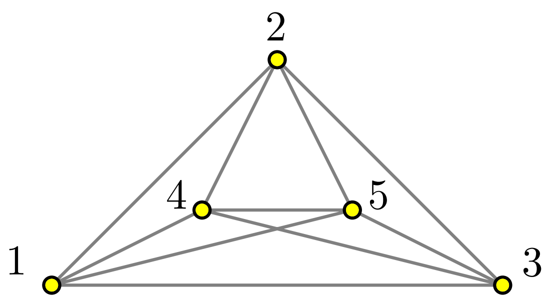
Lemma. Let G be a graph on n vertices where any 3 vertices span at least one edge. If $n \geq R_{3,3,9}$ then G contains 5 vertices such that for any two there exists a triple in which they span an edge.

We need a stronger statement where triples use different "dummy" vertices

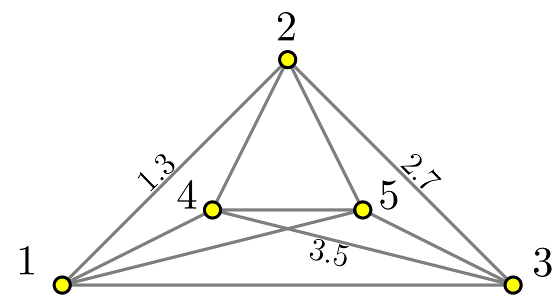
Let $I = \{i_1, i_2, \ldots, i_5\} \cup \{i_{1,2}, i_{1,3}, \ldots, i_{4,5}\}$.

Lemma. Let $\{a,b\} \in \binom{[3]}{2}$. There exists an injection from I into any ordered set of size ≥ 15 such that any $\{i_u, i_v\}$ are in $\{a,b\}$th position in $\{i_u, i_v, i_{u,v}\}$.

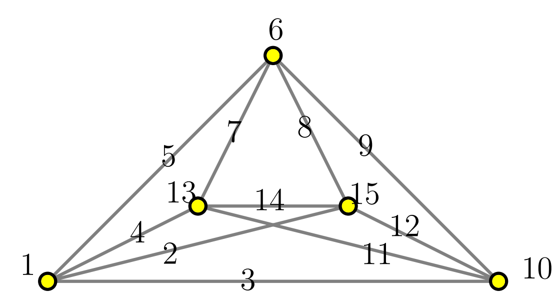
Proof:



Label the vertices of K_5 by $[5]$.



Given $\{a,b\}$, label the edges by distinct rationals such that in every "edge+vertices" triple the vertices are in positions a and b



map these labels to Z increasingly. □

$i_1, \ldots, i_5 =$ labels of the vertices

$i_{u,v} =$ label of the edge $i_u i_v$

Lemma. Let $\{a,b\} \in \binom{[3]}{2}$. There exists an injection from $\{i_1, i_2, \ldots, i_5\} \cup \{i_{1,2}, i_{1,3}, \ldots, i_{4,5}\}$ into any ordered set of size ≥ 15 such that any $\{i_u, i_v\}$ are in $\{a,b\}$th position in $\{i_u, i_v, i_{u,v}\}$.

Corollary. If $\mathcal{F}$ is a family of sets in $\mathbb{R}^2$ such that the intersection of any subfamily has at most two path-connected components then $\mathrm{Helly}(\mathcal{F}) \leq R_{3,3,15} - 1$.

Proof: Let $\mathcal{F} = \{A_1, A_2, \ldots A_k\}$ such that $k \geq R_{3,3,15}$ and $\forall j \leq k$, $\cap_{i \neq j} A_i \neq \emptyset$

Pick $p_j \in \cap_{i \neq j} A_i$

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Color $\{i_1, i_2, i_3\}$ with $i_1 < i_2 < i_3$ by a pair $\{a,b\}$ such that $i_a i_b$ is an edge in $\cap_{i \neq i_1, i_2, i_3} A_i$.

For $n \geq R_{3,3,15}$ some 15 vertices span triples all colored by the same pair $\{a,b\}$.

Lemma $\Rightarrow$ $i_1, \ldots, i_5$ and distinct $i_{u,v}$ for each $\{u,v\} \in \binom{[5]}{2}$

such that every $p_{i_u} p_{i_v}$ can be drawn in $\cap_{i \neq i_u, i_v, i_{u,v}} A_i$.

Two edges in this K_5 intersect and that intersection point lies in $\cap \mathcal{F}$. $\square$

The same idea works in higher dimension using that $\Delta_{2\lceil d/2\rceil+2}^{(\lceil d/2\rceil)} \not\hookrightarrow \mathbb{R}^d$.

Assuming intersections are k-connected, each "constrained" drawing of K_n extends into a "constrained" drawing of $\Delta_{n-1}^{(k)}$.

Every $p_{i_u}p_{i_v}$ is drawn in $\cap_{i\neq i_u,i_v,i_{u,v}} A_i$

Every $p_{i_u}p_{i_v}p_w$ is drawn in $\cap_{i\neq i_u,i_v,i_{u,v},i_w,i_{u,w},i_{v,w}} A_i$, etc...

Vertex-disjoint faces are drawn missing disjoint sets of A_i's

$\Rightarrow$ If $\mathcal{F}$ is a family of sets in $\mathbb{R}^d$ such that the intersection of any subfamily has at most 2 connected components, each $(\lceil d/2\rceil - 1)$-connected, then $\mathrm{Helly}(\mathcal{F}) \leq f(d)$.

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This was essentially the proof of:

Theorem. [Matoušek 1996] If $\mathcal{F}$ is a family of sets in $\mathbb{R}^d$ such that the intersection of any subfamily has at most r connected components, each $\lceil d/2\rceil$-connected, then $\mathrm{Helly}(\mathcal{F}) \leq f(r,d)$.

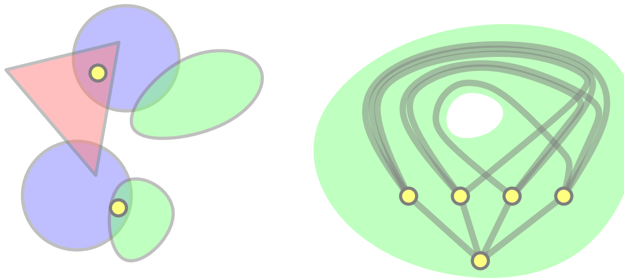
Uses a generalization of the "selection trick"

4. Bounds on Helly numbers arise from non-embeddability

5. Ramsey's theorem helps finding non-embeddable structures

6. Non-embeddability can be argued at the level of chain maps

$\tilde{\beta}_i(\cap\mathcal{G}) \leq b$ for all $\mathcal{G} \subseteq \mathcal{F}$ and $i \leq \lceil d/2 \rceil - 1$	$\Rightarrow$	$\mathrm{Helly}(\mathcal{F})$ is bounded by some function of d and b



Chain complex of a space or a simplicial complex.

$\oplus_n C_n$ where C_n is the $\mathbb{Z}_2$-vector space generated by the n-simplices

$\partial_n : C_n \to C_{n-1}$ are the boundary operators and satisfy $\partial_n \circ \partial_{n+1} = 0$

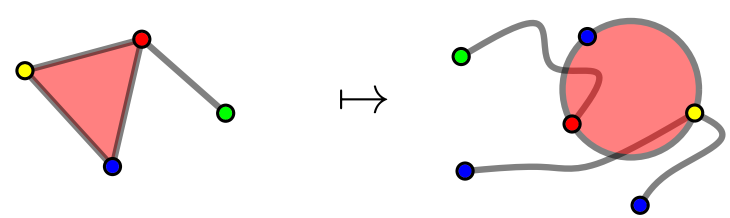
A chain map $\gamma : C_* \to D_*$ is a sequence of homomorphisms $\gamma_n : C_n \to D_n$ that commute with ∂.

$$\gamma_{n-1} \circ \partial_n^C = \partial_n^D \circ \gamma_n$$

K a simplicial complex and $\gamma : C_*(K) \to C_*(\mathbb{R}^d)$ a chain map.

γ is non-trivial if every vertex of K is mapped to a sum of an odd number of points.

γ is an homological almost embedding if it is non-trivial and for disjoint simplices $\sigma, \tau \in K$, $\gamma(\sigma)$ and $\gamma(\tau)$ have disjoint supports.



A continuous map $f : |K| \to \mathbb{R}^d$ induces a non-trivial chain map $f_\sharp : C_*(K) \to C_*(\mathbb{R}^d)$.

If f is an almost-embedding then $f_\sharp$ is an homological almost embedding.

Almost embedding for maps: disjoint simplices have disjoint images

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Theorem 5. There is no homological almost embedding from $C_*\left(\Delta_{d+1}^{(d)}\right)$ or from $C_*\left(\Delta_{d+2}^{(\lceil d/2 \rceil)}\right)$ into $C_*(\mathbb{R}^d)$.

Homological versions of the Radon and Van Kampen-Flores theorems

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Homological versions of the Radon and Van Kampen-Flores theorems

Proof shows that the Van Kampen obstruction to embeddability into $\mathbb{R}^d$ also forbids homological almost embeddings.

Technique: adapt the classical proof...

$\mathbb{Z}_2$ *spaces, equivariant maps, deleted products, Gauss map, Van Kampen obstruction*

... using equivariant chain homotopy [Wagner 2011]

Corollary. Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$. If for any $\mathcal{G} \subseteq \mathcal{F}$, $\cap\mathcal{G}$ is empty or has $\tilde{\beta}_i(\cap\mathcal{G}, \mathbb{Z}_2) = 0$ for $i = 0, 1, \ldots, \lceil d/2 \rceil - 1$ then $\mathrm{Helly}(\mathcal{F}) \leq d + 2$.

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Proof: Let $\mathcal{F} = \{A_1, A_2, \ldots A_k\}$ such that $k \geq d + 3$ and $\forall j \leq k$, $\cap_{i \neq j} A_i \neq \emptyset$

Construct a non-trivial chain map $\gamma : C_* \left(\Delta_{d+2}^{(\lceil d/2 \rceil)}\right) \to C_*(\mathbb{R}^d)$ "constrained by $\mathcal{F}$".

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$\tilde{\beta}_1(\cap_{i \neq u,v} A_i, \mathbb{Z}_2) = 0$ so $\gamma(\{u\}) + \gamma(\{v\})$ is a boundary.

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Inductive construction on $\Delta_{d+2}^{(\lceil d/2 \rceil)}$ then linear extension to $C_* \left(\Delta_{d+2}^{(\lceil d/2 \rceil)}\right)$.

If $\cap\mathcal{F} =$ then γ is an homological almost embedding. □

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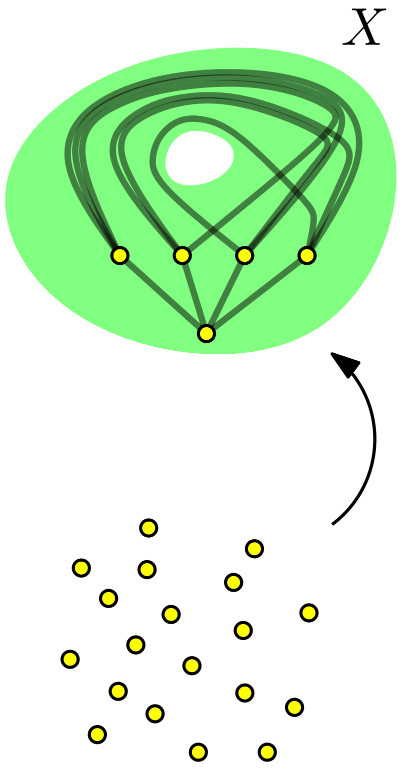
This simply repeats the previous homotopic arguments in a homological language.

Consider a chain map $\gamma : C_*(K_n) \to C_*(X)$ where X is an annulus.

X has two $\mathbb{Z}_2$-homology class in dimension 1.



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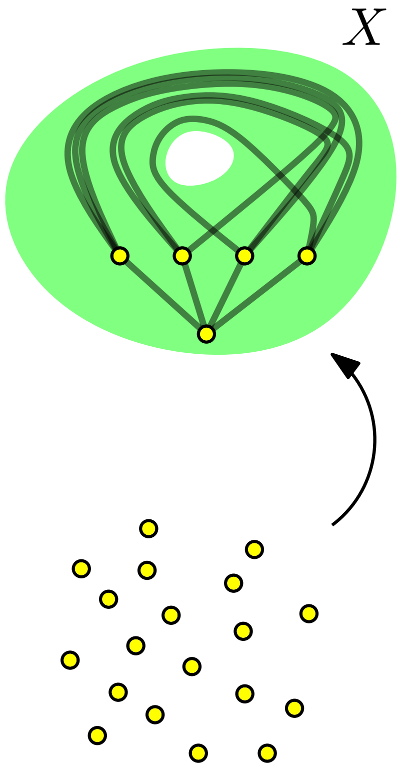

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Pick four vertices $v_1, v_2, v_3, v_4 \in K_n$

$$\partial v_1 v_2 v_3 + \partial v_1 v_2 v_4 = v_1 v_3 + v_3 v_2 + v_2 v_4 + v_4 v_1$$

so one of these three cycles is a boundary.

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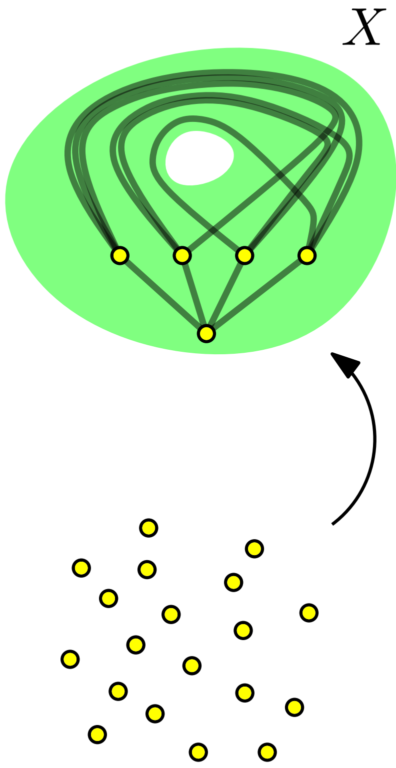

Lemma. Let $f : C_*(K_n) \to C_*(X)$ be a chain map and let $s \in \mathbb{N}$. For n large enough there exists a PL-embedding $g : K_s \to K_n$ such that for any $u, v, w \in K_s$, $f \circ g_\sharp(\partial uvw)$ is a boundary.

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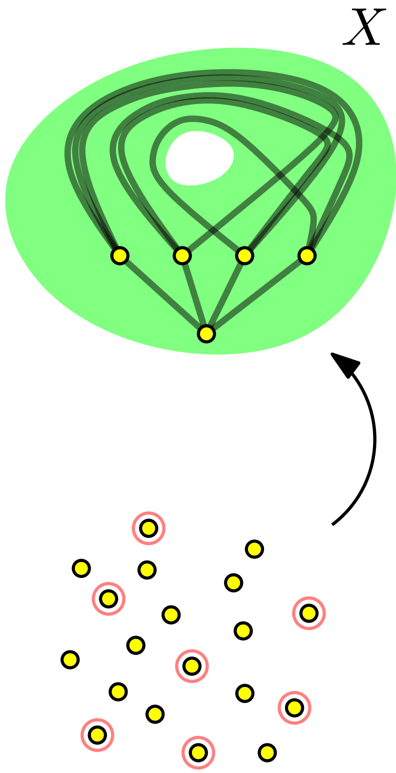

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Proof: Color every triangle xyz of K_n by the homology class of $\gamma(\partial xyz)$ in X. Use Ramsey's theorem to find t vertices so that all triangles have the same homology class under γ.

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$$\partial v_1 v_2 v_3 + \partial v_1 v_2 v_4 = v_1 v_3 + v_3 v_2 + v_2 v_4 + v_4 v_1$$

so one of these three cycles is a boundary.

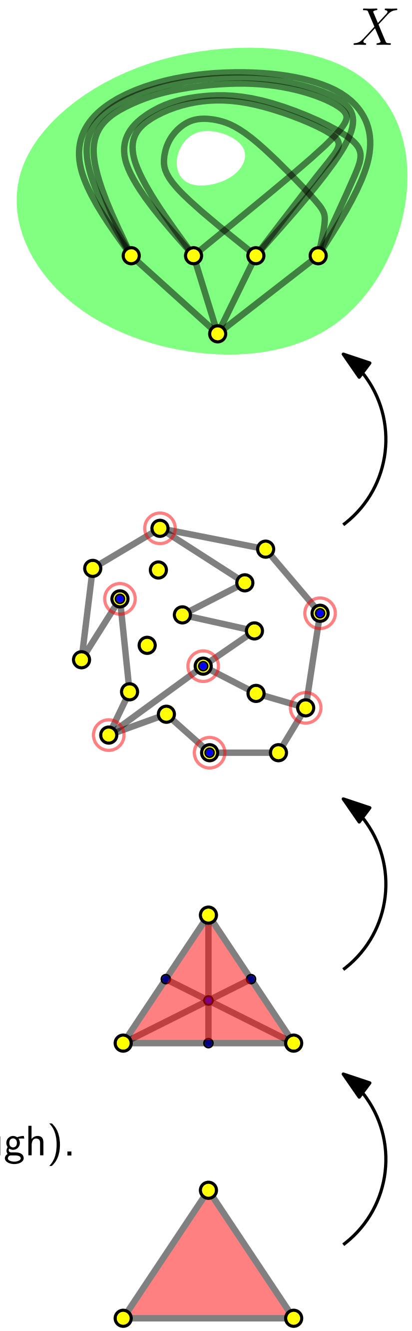
Lemma. Let $f : C_*(K_n) \to C_*(X)$ be a chain map and let $s \in \mathbb{N}$. For n large enough there exists a PL-embedding $g : K_s \to K_n$ such that for any $u, v, w \in K_s$, $f \circ g_\sharp(\partial uvw)$ is a boundary.

Proof: Color every triangle xyz of K_n by the homology class of $\gamma(\partial xyz)$ in X.

Use Ramsey's theorem to find t vertices so that all triangles have the same homology class under γ.

PL-map $\Delta^{(2)}_{s-1}$ to its barycentric subdivision sd $\Delta^{(2)}_{s-1}$.

Then map the 1-skeleton of sd $\Delta^{(2)}_{s-1}$ to these t vertices (assuming t is large enough).



Consider a chain map $\gamma : C_*(K_n) \to C_*(X)$ where X is an annulus.

X has two $\mathbb{Z}_2$-homology class in dimension 1.

Pick four vertices $v_1, v_2, v_3, v_4 \in K_n$

$$\partial v_1 v_2 v_3 + \partial v_1 v_2 v_4 = v_1 v_3 + v_3 v_2 + v_2 v_4 + v_4 v_1$$

so one of these three cycles is a boundary.

Lemma. Let $f : C_*(K_n) \to C_*(X)$ be a chain map and let $s \in \mathbb{N}$. For n large enough there exists a PL-embedding $g : K_s \to K_n$ such that for any $u, v, w \in K_s$, $f \circ g_\sharp(\partial uvw)$ is a boundary.

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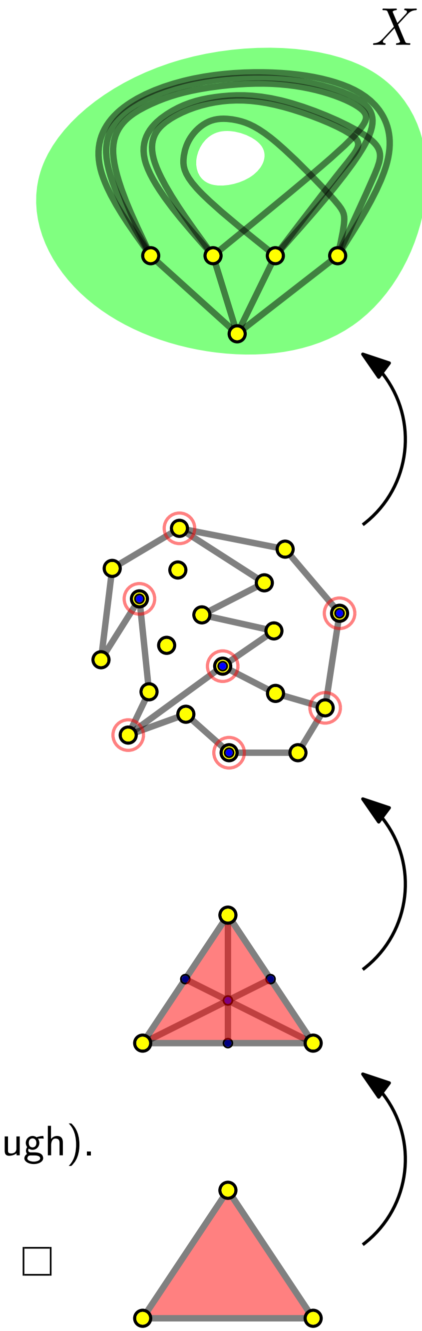
PL-map $\Delta^{(2)}_{s-1}$ to its barycentric subdivision sd $\Delta^{(2)}_{s-1}$.
Then map the 1-skeleton of sd $\Delta^{(2)}_{s-1}$ to these t vertices (assuming t is large enough).
Every triangle in K_s is the sum of 6 triangles in sd K_s.
A sum of an even number of times the same homology class is a $\mathbb{Z}_2$-boundary. $\square$



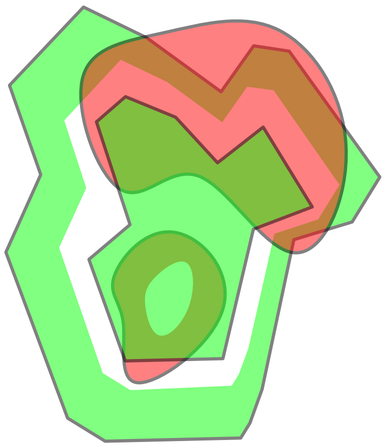
Applies in any dimension, provided the number of $\mathbb{Z}_2$-homology classes of the target space is bounded.

4. Bounds on Helly numbers arise from non-embeddability

5. Ramsey's theorem helps finding non-embeddable structures

6. Non-embeddability can be argued at the level of chain maps

$\tilde{\beta}_i(\cap\mathcal{G}) \leq b$ for all $\mathcal{G} \subseteq \mathcal{F}$ and $i \leq \lceil d/2 \rceil - 1$ $\Rightarrow$ $\mathrm{Helly}(\mathcal{F})$ is bounded by some function of d and b



Let $\mathcal{F} = \{A_1, A_2, \ldots A_k\}$ such that $\forall j \leq k$, $\cap_{i \neq j} A_i \neq \emptyset$

Assume that $\forall \mathcal{G} \subseteq \mathcal{F}$, $\cap \mathcal{G}$ has at most r connected components and $\tilde{\beta}_1(\cap \mathcal{G}, \mathbb{Z}_2) \leq r$

Goal: build a chain map from $C_*\left(\Delta_7^{(2)}\right)$ into $C_*(\mathbb{R}^d)$ such that disjoint faces σ, τ are mapped to chains supported in $\cap_{i \notin \Phi(\sigma)} A_i$ and $\cap_{i \notin \Phi(\tau)} A_i$ with $\Phi(\sigma) \cap \Phi(\tau) = \emptyset$

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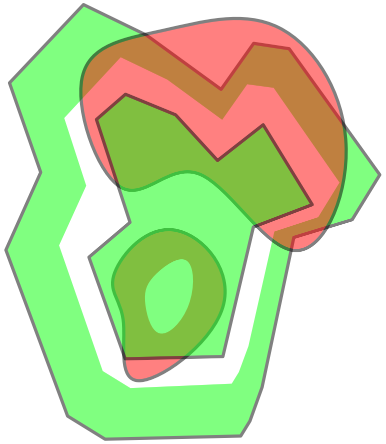

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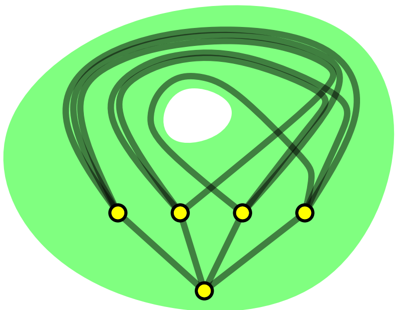

Ramsey $\rightsquigarrow T \subset [k]$ such that the positions of P_J for all $J \subset T$ are identical

Injection lemma $\rightsquigarrow$ a chain map $\gamma_1 : C_*(K_n) \to C_*\left(\cap_{i \notin T} A_i\right)$

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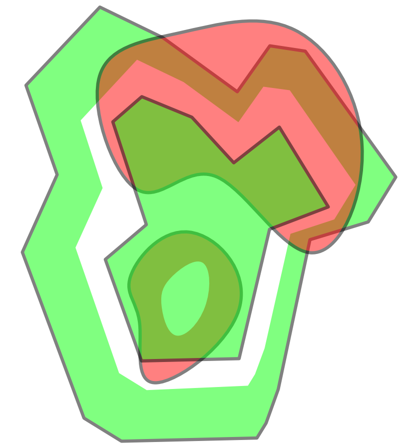

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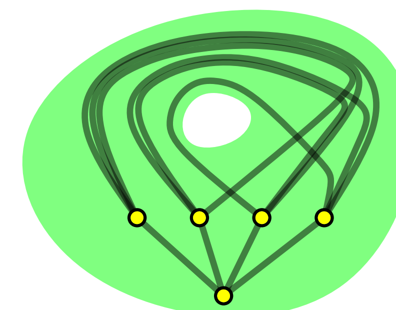

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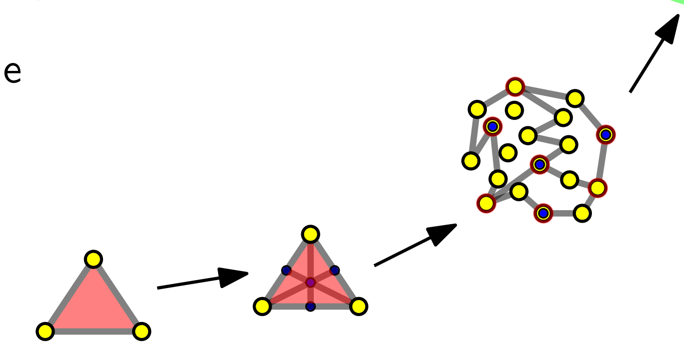

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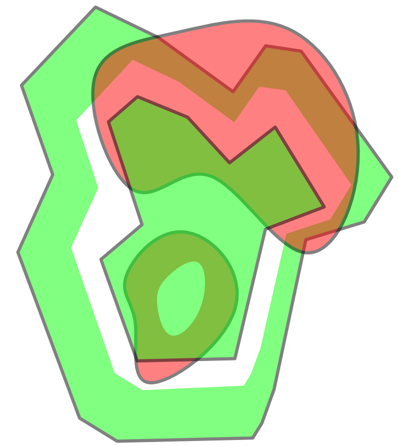

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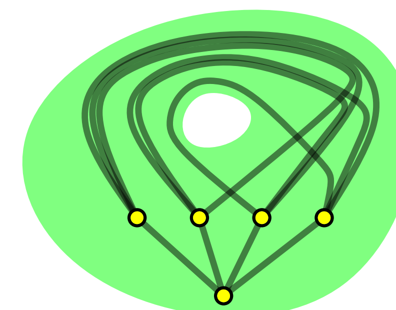

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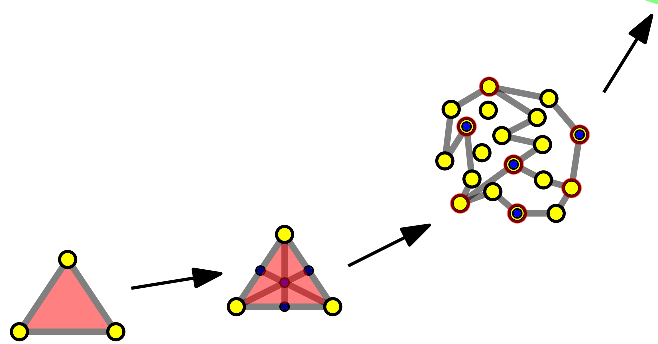

We can define $\gamma_2 : C_*(\Delta_{s-1}^{(2)}) \to C_*\left(\cap_{i \notin T} A_i\right)$

$n \geq R_{*,*,s^*}$, $|T| \geq R_{*,*,n^*}$ and $k \geq R_{*,*,|T|^*}$

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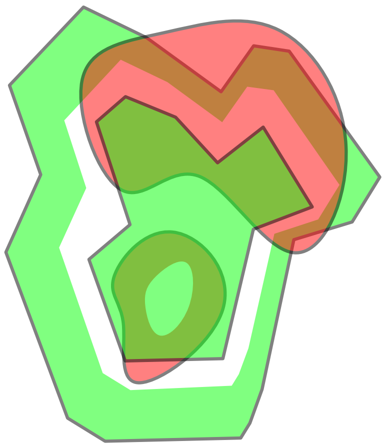

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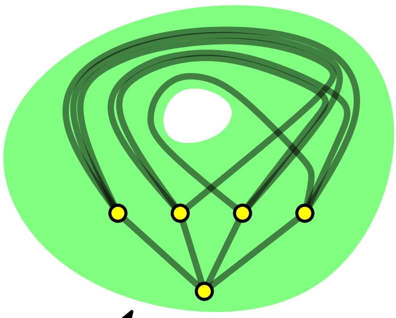

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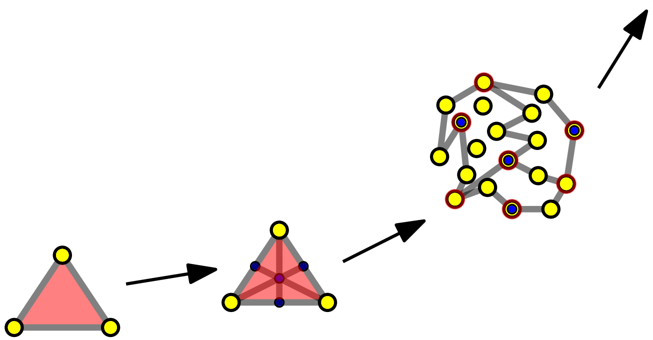

Problem: everything is supported in $\cap_{i \notin T} A_i$

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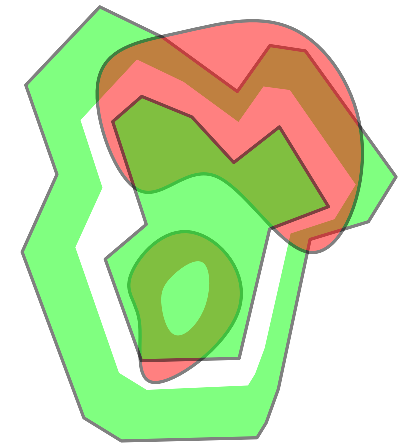

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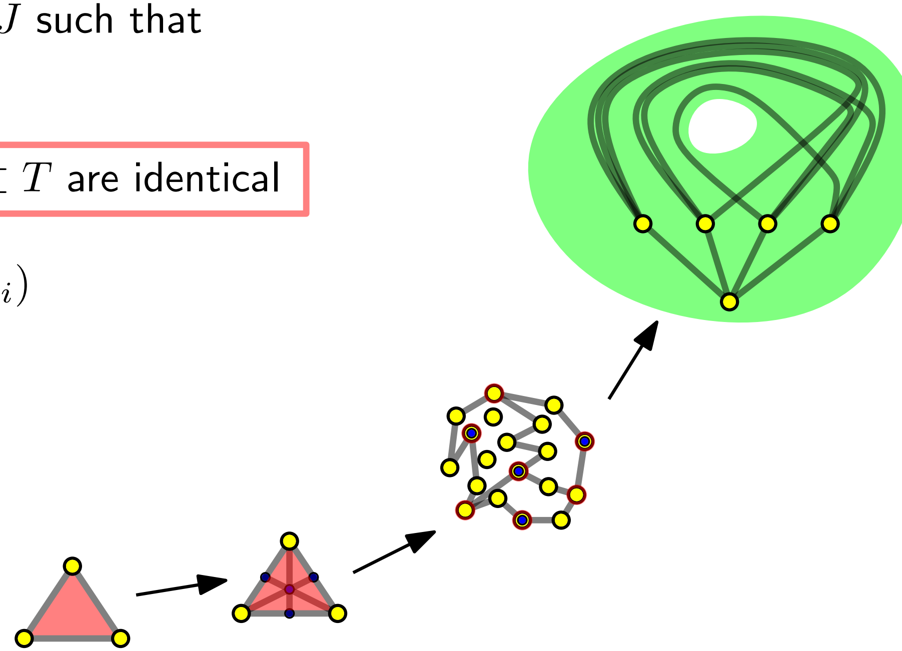

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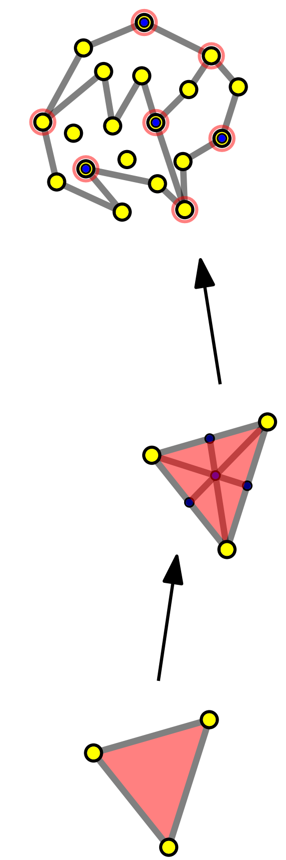
Pigeonhole $\rightsquigarrow$ any $(r+1)$-elements subset $J \subseteq [k]$ has a pair of points that forms a boundary in $\cap_{i \notin J} A_i$.

Color the $(r+1)$-uniform hypergraph on $[k]$ by the $\binom{r+1}{2}$ relative positions of these pairs.

Ramsey $\rightsquigarrow$ any set of size $m = R_{*,*,\ell^*}$ contains an ℓ-elements subset in which the relative positions are identical for all $(r+1)$-elements subsets.



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> **Injection lemma.** Let $\{a, b\} \in \binom{[3]}{2}$. There exists an injection from $\{i_1, i_2, \dots, i_5\} \cup \{i_{1,2}, i_{1,3}, \dots, i_{4,5}\}$ into any ordered set of size ≥ 15 such that any $\{i_u, i_v\}$ are in $\{a, b\}$th position in $\{i_u, i_v, i_{u,v}\}$.

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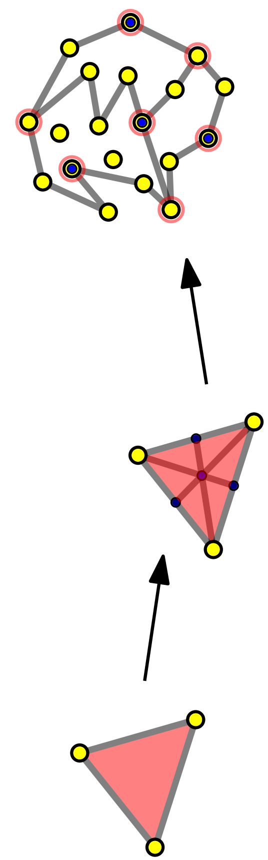

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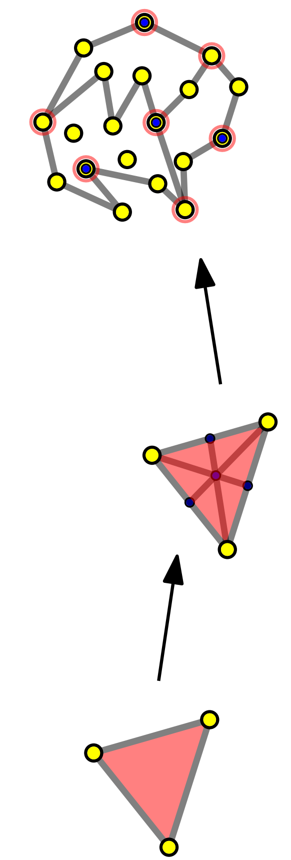

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$\Rightarrow$ We can extend γ over one (2D) triangle inside $\cap_{i \notin M} A_i$

Let $\mathcal{F} = \{A_1, A_2, \dots A_k\}$ such that $\forall j \leq k$, $\cap_{i \neq j} A_i \neq \emptyset$ and pick $p_j \in \cap_{i \neq j} A_i$ and define $\gamma(j) = p_j$

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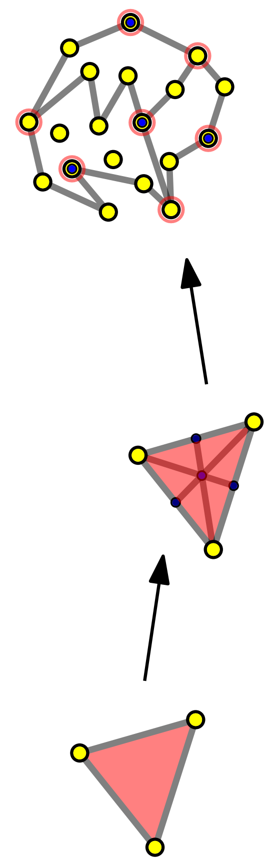
Recurse...



To summarize...

4. Bounds on Helly numbers arise from non-embeddability

Via embedding "constrained" by the intersection structure

Already hinted in the classical derivation of Helly from Radon

5. Ramsey's theorem helps finding non-embeddable structures

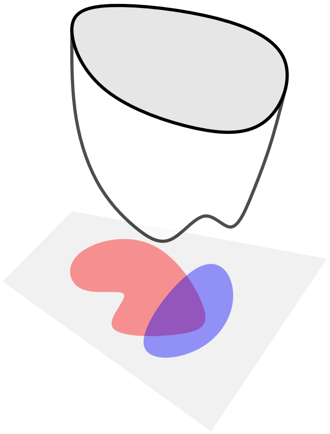
Uniform "r in ℓ" selection

6. Non-embeddability can be argued at the level of chain maps

Classical proofs carry from almost embedding to homological almost embeddings

This makes finding boundaries much easier (mod 2)

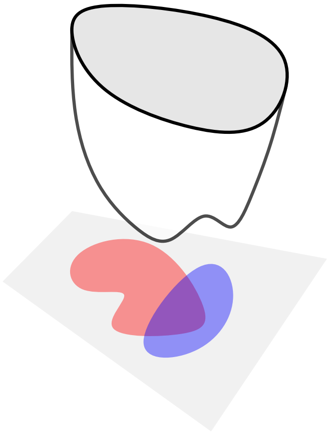
$\tilde{\beta}_i(\cap\mathcal{G}) \leq b$ for all $\mathcal{G} \subseteq \mathcal{F}$ and $i \leq \lceil d/2 \rceil - 1$ $\Rightarrow$ $\mathrm{Helly}(\mathcal{F})$ is bounded by some function of d and b



A consequence on the complexity of optimization problems

$\min_{\cap_i C_i} f$

where $f : \mathbb{R}^d \to \mathbb{R}$ and $C_1, C_2, \ldots, C_n$ subsets of $\mathbb{R}^d$



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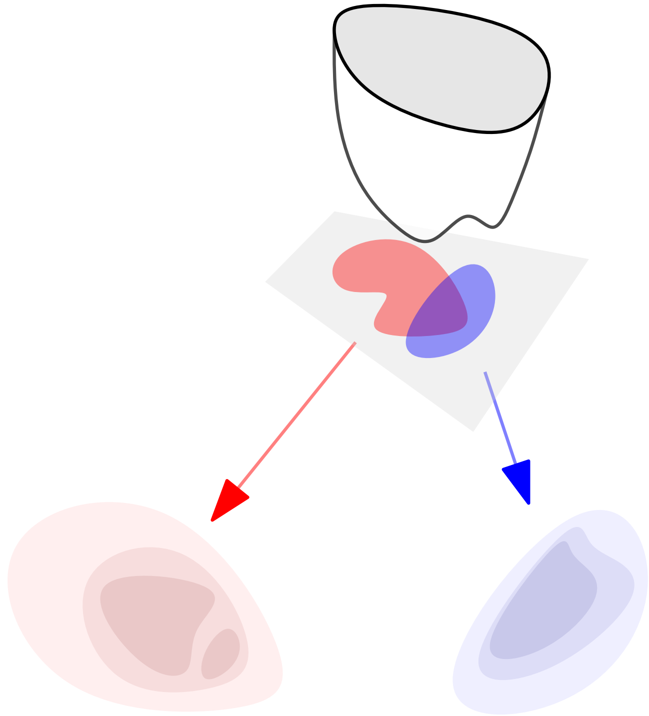

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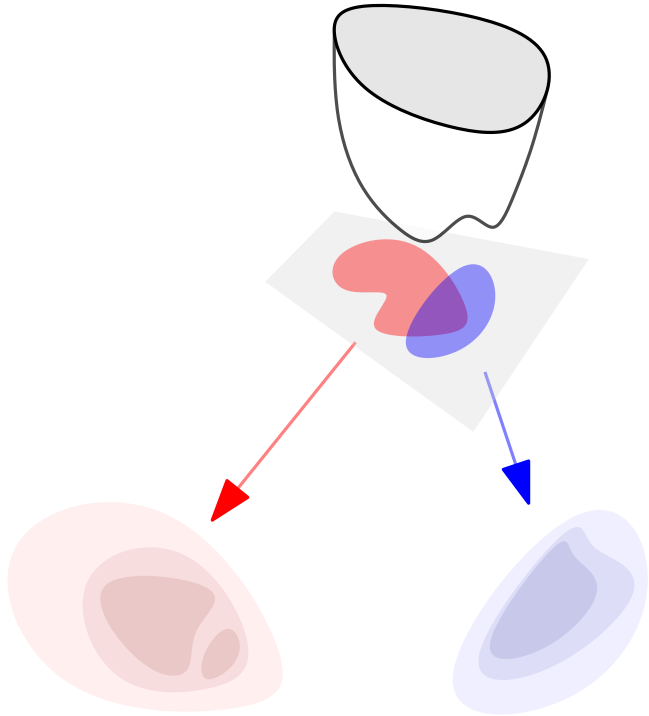
If the maximum Helly number of the sets $\{C_1, C_2, \ldots, C_n, f^{-1}((-\infty, t))\}$ is some constant h (indpt of n) then there exists $i_1, i_2, \ldots, i_h$ such that

$$\min_{\cap_i C_i} f = \min_{\cap_{j=1}^h C_{i_j}} f$$

and they can be computed “efficiently”.

A consequence on the complexity of optimization problems



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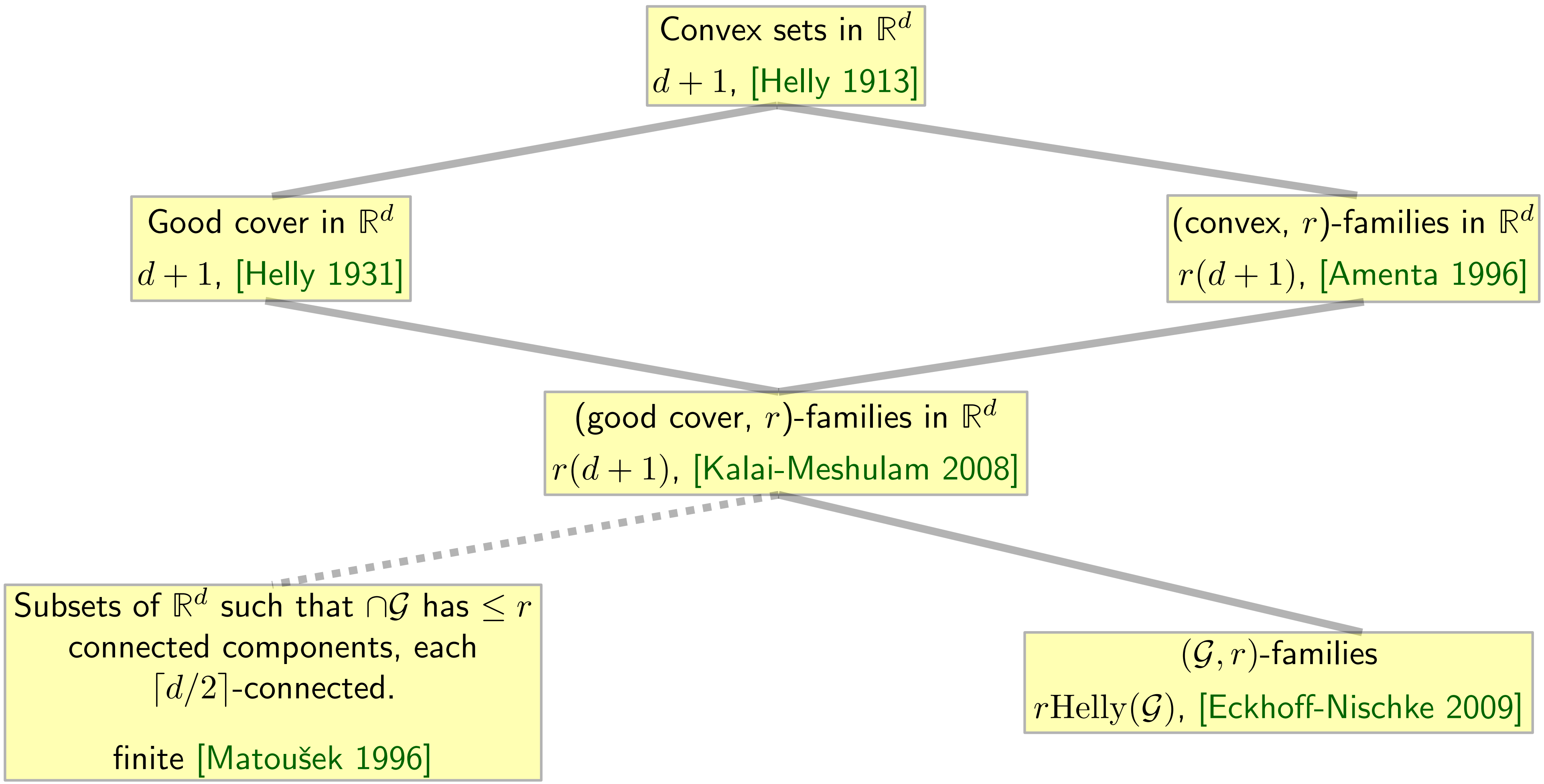
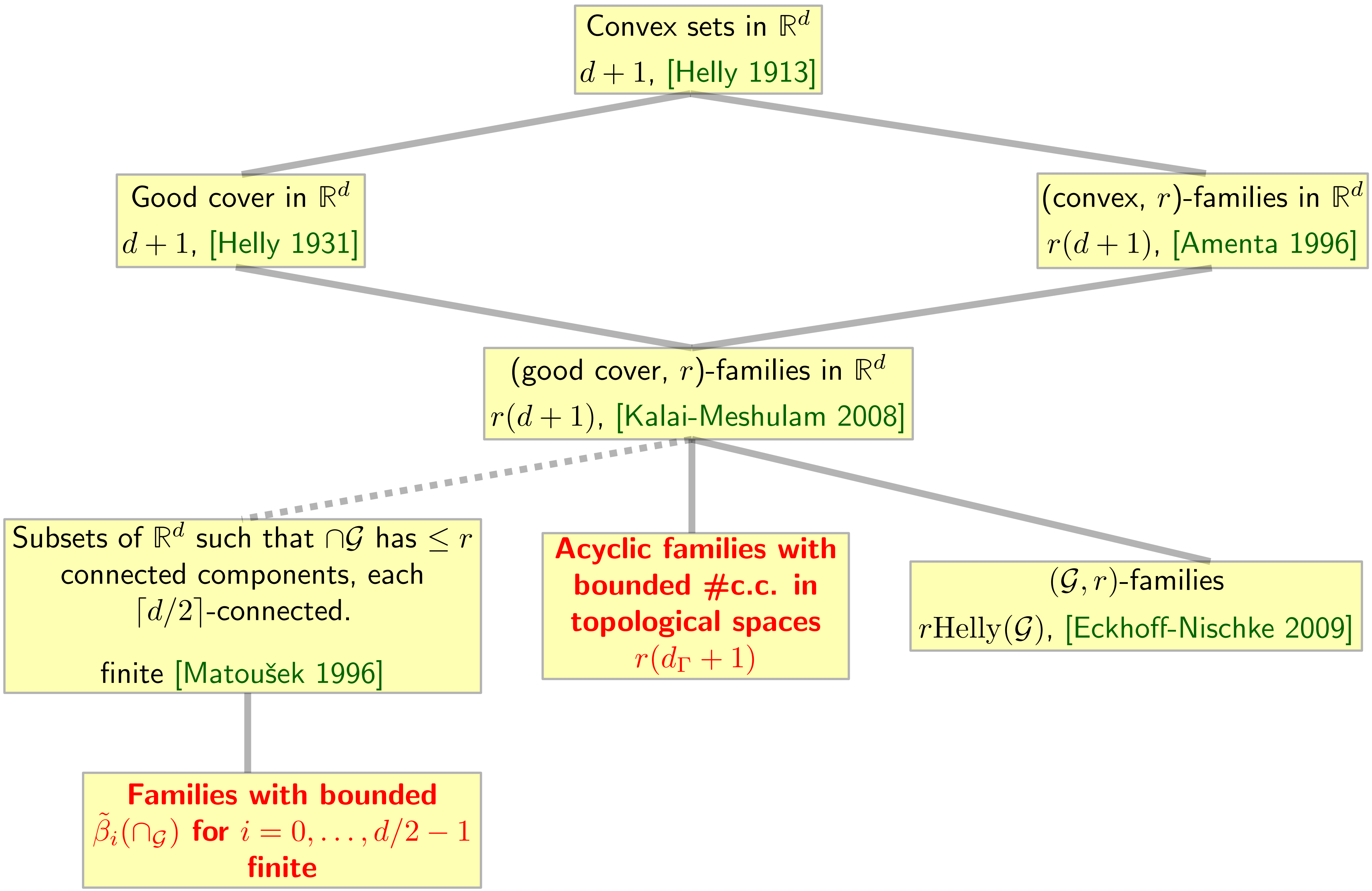
For this number not to be bounded requires “unbounded topological complexity” in the level sets of the C_i.

Perspectives



$\mathcal{F}$ is a $(r, \mathcal{G})$-family if every intersection of members of $\mathcal{F}$ is a disjoint union of at most r members of $\mathcal{G}$.



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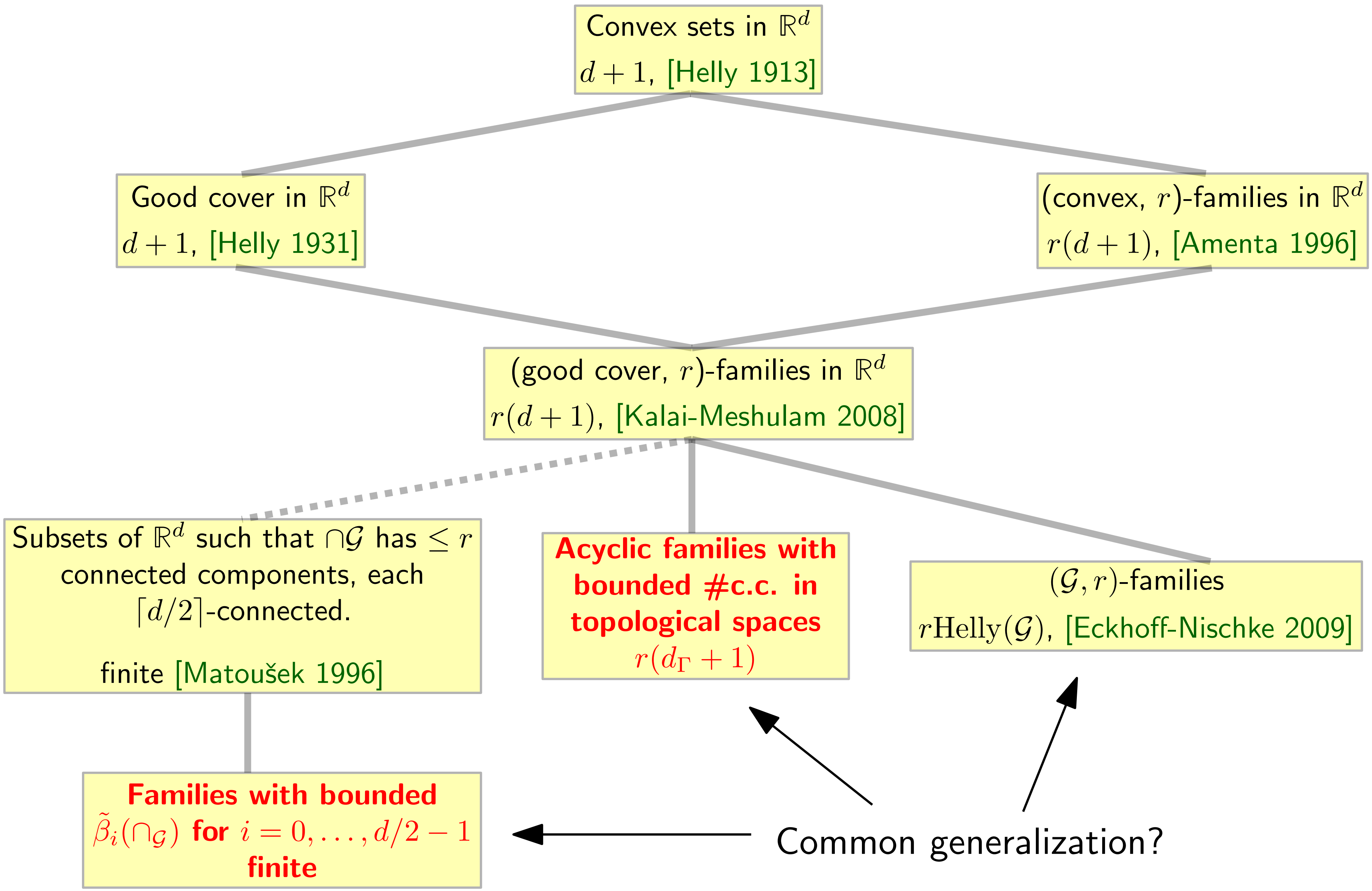


$\mathcal{F}$ is a $(r, \mathcal{G})$-family if every intersection of members of $\mathcal{F}$ is a disjoint union of at most r members of $\mathcal{G}$.

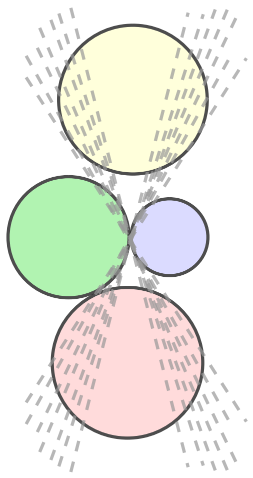
Helly numbers of sets of line transversals to

disjoint unit disks in $\mathbb{R}^2$: ≤ 5 [Danzer 1957]

disjoint translates of a convex figure in $\mathbb{R}^2$: ≤ 5 [Tverberg 1989]

disjoint translates of a convex polyhedron in $\mathbb{R}^3$: unbounded [Holmsen-Matoušek 2004]

disjoint unit balls in $\mathbb{R}^d$: $\leq 4d-1$ [Cheong-Holmsen-G-Petitjean 2006]



Could we also obtain

Hadwiger's transversal theorem. Let $C_1, C_2, \ldots C_n$ be disjoint convex sets in the plane. If any three have an oriented line transversal in increasing order then they all have a line transversal.

from topological arguments?

Let X and Y be simplicial complexes.

Let $\pi : X \to Y$ be a surjective, dimension preserving, $\leq r$-to-one simplicial map.

Theorem. [Kalai-Meshulam 2008] $L(Y) + 1 \leq r(L(X) + 1)$.

Theorem. [Eckhoff-Nishke 2009] $H(Y) \leq rH(X)$.

Theorem. [Amenta 1996] $\Delta(Y) + 1 \leq r(\Delta(X) + 1)$.

Is there some common generalization?

A simplicial hole is an induced subcomplex isomorphic to the boundary of a simplex.
Define $H(K)$ as the maximum dimension of a simplicial hole of K.

$\Delta(K) \simeq$ the collapsibility of K.

Thank you for your attention