High Dimensional Expansion

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Plan

Cohomological Expansion

- The k-dimensional Cheeger constant
- Homology of random complexes
- Expansion of symmetric complexes
- Expansion and topological overlap
- 2-expanders from random Latin squares

Expansion via Eigenvalues

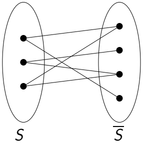
- Spectral gap of the k-Laplacian
- Spectral gap and colored simplices
- Garland's method
- Homology of random flag complexes
- Spectral gap and hypergraph matching

Graphical Cheeger Constant

Edge Cuts

For a graph $G = (V, E)$ and $S \subset V$, $\overline{S} = V - S$ let

$$e(S, \overline{S}) = |\{e \in E : |e \cap S| = 1\}|.$$



Cheeger Constant

$$h(G) = \min_{0<|S|\leq \frac{|V|}{2}} \frac{e(S, \overline{S})}{|S|}.$$

Graphical Spectral Gap

Laplacian Matrix

$G = (V, E)$ a graph, $|V| = n$.

The Laplacian of G is the $V \times V$ matrix L_G:

$$L_G(u, v) = \begin{cases} \deg(u) & u = v \\ -1 & uv \in E \\ 0 & \text{otherwise.} \end{cases}$$

Eigenvalues of L_G

$$0 = \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G).$$

$$\lambda_2(G) = \text{Spectral Gap of } G.$$

Cheeger Constant vs. Spectral Gap

Theorem [Alon-Milman, Tanner]:

For all $\emptyset \neq S \subsetneq V$

$$e(S, \overline{S}) \geq \frac{|S||\overline{S}|}{n} \lambda_2(G).$$

In particular

$$h(G) \geq \frac{\lambda_2(G)}{2}.$$

Theorem [Alon, Dodziuk]:

If G is d-regular then

$$h(G) \leq \sqrt{2d\lambda_2(G)}.$$

$h(G)$ and $\lambda_2(G)$ are therefore essentially equivalent measures of graphical expansion.

High Dimensional Expansion

The notions of Cheeger Constant and Spectral Gap have natural high dimensional extensions. They are however not equivalent in dimensions greater than one.

Cohomological Expansion

- Linial-M-Wallach: Homology of random complexes.
- Gromov: The topological overlap property.
- Gundert-Wagner: Expansion of random complexes.

Spectral Gap of k-Laplacians

- Garland: Cohomology of discrete groups.
- Aharoni-Berger-M: Hypergraph matching.
- Kahle: Homology of random flag complexes.

Simplicial Cohomology

X a simplicial complex on V, R a fixed abelian group.
i-face of $\sigma = [v_0, \cdots, v_k]$ is $\sigma_i = [v_0, \cdots, \widehat{v_i}, \cdots, v_k]$.
$C^k(X) = k$-cochains $=$ skew-symmetric maps $\phi : X(k) \rightarrow R$.
Coboundary Operator $d_k : C^k(X) \rightarrow C^{k+1}(X)$ given by

$$d_k\phi(\sigma) = \sum_{i=0}^{k+1} (-1)^i \phi(\sigma_i) \quad .$$

$d_{-1} : C^{-1}(X) = R \rightarrow C^0(X)$ given by
$d_{-1}a(v) = a$ for $a \in R$, $v \in V$.
$Z^k(X) =$ k-cocycles $= \ker(d_k)$.
$B^k(X) =$ k-coboundaries $= \mathrm{Im}(d_{k-1})$.
k-th reduced cohomology group of X:

$$\tilde{\mathrm{H}}^k(X) = \tilde{\mathrm{H}}^k(X; R) = Z^k(X)/B^k(X) \quad .$$

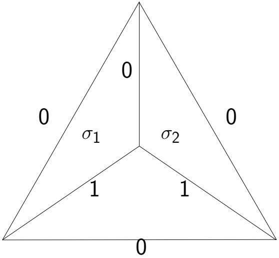
Cut of a Cochain

Cut determined by a k-cochain $\phi \in C^k(X;R)$:

$$\mathrm{supp}(d_k\phi) = \{\tau \in X(k+1) \ : \ d_k\phi(\tau) \neq 0\} \ .$$

Cut Size of ϕ: $\|d_k\phi\| = |\mathrm{supp}(d_k\phi)|$.

Example:



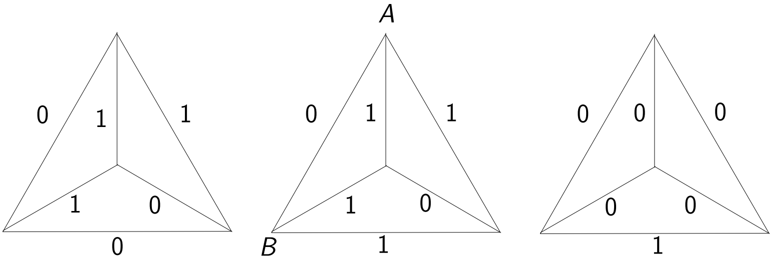
$$\|d_1\phi\| = |\{\sigma_1, \sigma_2\}| = 2$$

Hamming Weight of a Cochain

The Weight of a k-cochain $\phi \in C^k(X;R)$:

$$\|[\phi]\| = \min \ \{ \ |\mathrm{supp}(\phi + d_{k-1}\psi)| \ : \ \psi \in C^{k-1}(X;R) \ \}.$$

Example: $\|\phi\| = 3$ but $\|[\phi]\| = 1$



ϕ $\quad$ $d_0 1_{A,B}$ $\quad$ $\phi - d_0 1_{A,B}$

Expansion of a Complex

Expansion of a Cochain

The expansion of $\phi \in C^k(X;R) - B^k(X;R)$ is

$$\frac{\|d_k \phi\|}{\|[\phi]\|}.$$

k-expansion Constant

$$h_k(X;R) = \min\left\{ \frac{\|d_k \phi\|}{\|[\phi]\|} : \phi \in C^k(X;R) - B^k(X;R) \right\}.$$

Remarks:

- G graph $\Rightarrow$ $h_0(G;\mathbb{F}_2) = h(G)$.
- $h_k(X;R) > 0 \Leftrightarrow \tilde{H}^k(X;R) = 0$.
- In the sequel: $h_k(X) = h_k(X;\mathbb{F}_2)$.

Expansion of a Simplex I

Δ_{n-1} = the $(n-1)$-dimensional simplex on $V = [n]$.

Claim [M-Wallach, Gromov]:

$$h_{k-1}(\Delta_{n-1}) = \frac{n}{k+1}.$$

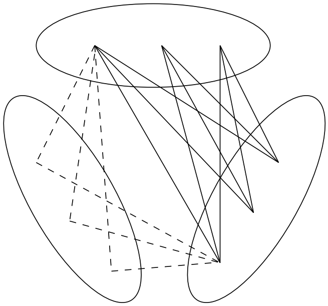
Example:

$[n] = \bigcup_{i=0}^{k} V_i \ , \ |V_i| = \frac{n}{k+1}$

$\phi = 1_{V_0 \times \cdots \times V_{k-1}}$

$\|[\phi]\| = (\frac{n}{k+1})^k$

$\|d_{k-1}\phi\| = (\frac{n}{k+1})^{k+1}$



Expansion of a Simplex II

Let $\phi \in C^{k-1}(\Delta_{n-1})$. For $u \in V$ define $\phi_u \in C^{k-2}(\Delta_{n-1})$ by

$$\phi_u(\tau) = \begin{cases} \phi(u\tau) & u \notin \tau \\ 0 & u \in \tau \end{cases} .$$

Let $\sigma \in \Delta_{n-1}(k-1)$ and $u \in V$. Then:

$$d_{k-1}\phi(u\sigma) = \phi(\sigma) - \sum_{w \in \sigma} \phi(u(\sigma - w)) = \phi(\sigma) - d_{k-2}\phi_u(\sigma).$$

Therefore:

$$\begin{aligned}
&(k+1)\|d_{k-1}\phi\| \\
&\qquad = |\{(\tau, u) \in \Delta_{n-1}(k) \times V : u \in \tau \in \mathrm{supp}(d_{k-1}\phi)\}| \\
&\qquad = |\{(\sigma, u) \in \Delta_{n-1}(k-1) \times V : \sigma \in \mathrm{supp}(\phi - d_{k-2}\phi_u)\}| \\
&\qquad = \sum_{u \in V} |\mathrm{supp}(\phi - d_{k-2}\phi_u)| \geq n\|[\phi]\|.
\end{aligned}$$

Topology of a Random Graph

Theorem [Erdős-Rényi '58]:

For any function $\omega(n)$ that tends to infinity

$$\lim_{n\to\infty} \Pr \left[\ G \in G(n,p) \ : \ G \ \text{ connected } \right] =$$

$$\begin{cases} 0 & p = \frac{\log n - \omega(n)}{n} \\ 1 & p = \frac{\log n + \omega(n)}{n} \end{cases} .$$

$$\lim_{n\to\infty} \Pr \left[\ G \in G(n,\frac{c}{n}) \ : \ G \ \text{ acyclic } \right] =$$

$$\begin{cases} 0 & c > 1 \\ \sqrt{1-c} \cdot e^{\frac{2c+c^2}{4}} & c < 1 \end{cases} .$$

A Model of Random Complexes

Y a simplicial complex , $Y^{(i)} = i$-dim skeleton of Y.
$Y(i) =$ oriented i-dim simplices of Y.
$f_i(Y) = |Y(i)|$.
$\Delta_{n-1} =$ the $(n-1)$-dimensional simplex on $V = [n]$.

$Y_k(n,p) =$ probability space of all complexes

$$\Delta_{n-1}^{(k-1)} \subset Y \subset \Delta_{n-1}^{(k)}$$

with probability distribution

$$\Pr(Y) = p^{f_k(Y)}(1-p)^{\binom{n}{k+1} - f_k(Y)} \ .$$

Homological Connectivity of Random Complexes

Fix $k \geq 1$ and a finite abelian group R.

Theorem [Linial-M '03 , M-Wallach '06]:

For any function $\omega(n)$ that tends to infinity

$$\lim_{n\to\infty} \Pr\left[\, Y \in Y_k(n,p) : \tilde{\mathrm{H}}_{k-1}(Y;R) = 0 \,\right] = \begin{cases} 0 & p = \frac{k\log n - \omega(n)}{n} \\ 1 & p = \frac{k\log n + \omega(n)}{n} \end{cases} .$$

Expansion and Homological Connectivity

A weak threshold:

If $p = \frac{(k^2+k+1)\log n}{n}$ then a.a.s. $H^{k-1}(Y;\mathbb{F}_2) = 0$.

Proof:

$$\Pr\ [\ \tilde{\mathrm{H}}^{k-1}(Y;\mathbb{F}_2) \neq 0\ ]$$

$$\leq \sum_{0\neq[\phi]\in\tilde{\mathrm{H}}^{k-1}(\Delta_{n-1}^{(k-1)})} (1-p)^{\|d_{k-1}\phi\|}$$

$$\leq \sum_{m\geq 1} \binom{\binom{n}{k}}{m} (1-p)^{\frac{nm}{k+1}}$$

$$\leq \sum_{m\geq 1} (n^k n^{-\frac{k^2+k+1}{k+1}})^m = \sum_{m\geq 1} (n^{-\frac{1}{k+1}})^m \longrightarrow 0.$$

Weighted Expansion

X - n-dimensional pure simplicial complex.
A probability distribution on $X(k)$:

$$w(\sigma) = \frac{|\{\eta \in X(n) : \sigma \subset \eta\}|}{\binom{n+1}{k+1} f_n(X)}.$$

For $\phi \in C^k(X)$ let

$$\|\phi\|_w = \sum_{\{\sigma \in X(k) : \phi(\sigma) \neq 0\}} w(\sigma)$$

$$\|[\phi]\|_w = \min\{\|\phi + d_{k-1}\psi\|_w : \psi \in C^{k-1}(X)\}.$$

Weighted k-th Expansion:

$$\underline{h}_k(X) = \min\left\{\frac{\|d_k\phi\|_w}{\|[\phi]\|_w} : \phi \in C^k(X) - B^k(X)\right\}.$$

Building-Like Complexes

X - n-complex, $G < Aut(X)$, S - a G-set.
G acts diagonally on $\mathcal{F}_k = S \times X(k)$.
A family of "apartment-like" subcomplexes of X:

$$\mathcal{B} = \{B_{s,\tau} : -1 \leq k < n, (s,\tau) \in \mathcal{F}_k\}$$

such that $\tau \in B_{s,\tau} \subset B_{s,\tau'} \quad \forall\, s \in S \;,\; \tau \subset \tau' \in X^{(n-1)}$.

Building-like complex

4-tuple $(X, S, G, \mathcal{B})$ satisfying:

- G is transitive on $X(n)$.
- $gB_{s,\tau} = B_{gs,g\tau}$ for all $g \in G$ and $(s,\tau) \in S \times X^{(n-1)}$.
- $\tilde{H}_i(B_{s,\tau}) = 0$ for all $(s,\tau) \in \mathcal{F}_k$ and $-1 \leq i \leq k < n$.

Example: Symmetric Matroids

Matroid:

An n-dimensional simplicial complex $M \subset 2^V$ such that $M[S]$ is pure for all $S \subset V$.

Homology of matroids:

$\tilde{H}_i(M) = 0$ for all $0 \leq i \leq \dim M - 1$.

Symmetric matroid:

$G = Aut(M)$ is transitive on the maximal faces.

Symmetric matroids as building-like complexes:

$S = M(n)$ and for $(s, \tau) \in S \times M(k) = \mathcal{F}_k$

$$B_{s,\tau} = M[s \cup \tau].$$

Example: The Spherical Buildings $\Delta = A_{n+1}(\mathbb{F}_q)$

Vertices: All nontrivial linear subspaces $0 \neq V \subsetneq \mathbb{F}_q^{n+2}$.
Simplices: $V_0 \subset \cdots \subset V_k$.

Homology of Δ [Solomon, Tits]:
$\tilde{\mathrm{H}}_i(\Delta) = 0$ for $i < n$ and $\dim \tilde{\mathrm{H}}_n(\Delta) = q^{\binom{n+2}{2}}$.

Standard apartment A:
$e_1, \ldots, e_{n+2}$ standard basis of $\mathbb{F}_q^{n+2}$.
$A = \Delta[\{\langle e_{i_0}, \ldots, e_{i_k} \rangle : 1 \leq i_0 < \cdots < i_k \leq n+2\}] \cong S^n$

Δ as a building-like complex:
$S = \Delta(n)$ and for $(s, \tau) \in S \times \Delta(k) = \mathcal{F}_k$

$$B_{s,\tau} = \cap\{gA : g \in GL_{n+2}(\mathbb{F}_q), \quad s, \tau \in gA\}.$$

Expansion of Building-Like Complexes

$(X, S, G, \mathcal{B})$ - n-dimensional building-like complex.
For $0 \le k \le n-1$, let

$$a_k = \max\{|G\eta \cap B_{s,\tau}(k+1)| : \eta \in X(k+1), (s,\tau) \in \mathcal{F}_k\}.$$

The following result is inspired by work of Gromov.

Theorem [Lubotzky-M-Mozes]:

$$\underline{h}_k(X) \ge \left(\binom{n+1}{k+2} a_k\right)^{-1}.$$

Expansion of Symmetric Matroids

Proposition [LMM]:

M symmetric matroid $\;\Rightarrow\;$ $\underline{h}_k(M) \geq 8^{-\dim M}$ $\quad \forall\, k \leq \dim M - 1$.

Example: The Partition Matroid $X_{n,m}$

Let $V_1, \ldots, V_{n+1}$ be $n+1$ disjoint sets, $|V_i| = m$.
$\sigma \in X_{n,m}$ iff $|\sigma \cap V_i| \leq 1$ for all $1 \leq i \leq n+1$.

Proposition [LMM]:

For $0 \leq k \leq n-1$

$$\underline{h}_k(X_{n,m}) \geq \frac{\binom{n+1}{k+1}}{\sum_{j=0}^{k+1} \left(\frac{2(m-1)}{m}\right)^j \binom{n-j}{n-k-1}}.$$

In particular, $\underline{h}_k(X_{n,2}) \geq 1$ and

$$\underline{h}_{n-1}(X_{n,m}) \geq \frac{n+1}{\sum_{j=0}^{n} \left(\frac{2(m-1)}{m}\right)^j} > \frac{n+1}{2^{n+1}-1}.$$

Expansion of Spherical Buildings

The Building $A_{n+1}(\mathbb{F}_q)$

Vertices: All nontrivial linear subspaces $0 \neq V \subsetneq \mathbb{F}_q^{n+2}$.
Simplices: $V_0 \subset \cdots \subset V_k$.

Example: The Projective Plane Graph

$A_2(\mathbb{F}_q)$ - Points vs. Lines bipartite graph of $PG(2,q)$.
$\underline{h}_0(A_2(\mathbb{F}_q)) = 1 - o(1)$ as $q \to \infty$.

Proposition [Gromov, LMM]:

$$\underline{h}_{n-1}(A_{n+1}(\mathbb{F}_q)) \geq \frac{1}{(n+2)!}.$$

Problem: Determine

$$\lim_{q\to\infty} \underline{h}_{n-1}(A_{n+1}(\mathbb{F}_q)).$$

The Affine Overlap Property

Number of Intersecting Simplices

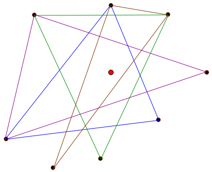
For $A = \{a_1, \ldots, a_n\} \subset \mathbb{R}^k$ and $p \in \mathbb{R}^k$ let

$$\gamma_A(p) = |\{\sigma \subset [n] : |\sigma| = k+1 \ , \ p \in \text{conv}\{a_i\}_{i \in \sigma}\}|.$$

Theorem [Bárány]:

There exists $p \in \mathbb{R}^k$ such that

$$f_A(p) \geq \frac{1}{(k+1)^k} \binom{n}{k+1} - O(n^k).$$



The Topological Overlap Property

Number of Intersecting Images

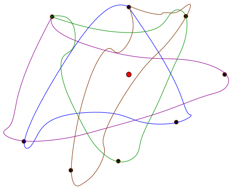
For a continuous map $f : \Delta_{n-1} \to \mathbb{R}^k$ and $p \in \mathbb{R}^k$ let

$$\gamma_f(p) = |\{\sigma \in \Delta_{n-1}(k) : p \in f(\sigma)\}|.$$

Theorem [Gromov]:

There exists $p \in \mathbb{R}^k$ such that

$$\gamma_f(p) \geq \frac{2k}{(k+1)!(k+1)}\binom{n}{k+1} - O(n^k).$$



Topological Overlap and Expansion

Number of Intersecting Images

For a continuous map $f : X \to \mathbb{R}^k$ and $p \in \mathbb{R}^k$ let

$$\gamma_f(p) = |\{\sigma \in X(k) : p \in f(\sigma)\}|.$$

Expansion Condition on X

Suppose that for all $0 \leq i \leq k-1$

$$\underline{h}_i(X) \geq \epsilon.$$

Theorem [Gromov]

There exists a $\delta = \delta(k, \epsilon)$ such that for any continuous map $f : X \to \mathbb{R}^k$ there exists a $p \in \mathbb{R}^k$ such that

$$\gamma_f(p) \geq \delta f_k(X).$$

Expander Graphs

(d, ϵ)-Expanders

A family of graphs $\{G_n = (V_n, E_n)\}_n$ with $|V_n| \to \infty$ with two seemingly contradicting properties:

- High Connectivity: $h(G_n) \geq \epsilon$.
- Sparsity: $\max_v \deg_{G_n}(v) \leq d$.

Pinsker:

Random $3 \leq d$-regular graphs are (d, ϵ)-expanders.

Margulis:

Explicit construction of expanders.

Lubotzky-Phillips-Sarnak, Margulis:

Ramanujan Graphs - an "optimal" family of expanders.

Expander Complexes

Degree of a Simplex

For $\sigma \in X(k-1)$ let $\deg(\sigma) = |\{\tau \in X(k) : \sigma \subset \tau\}|$.
$D_{k-1}(X) = \max_{\sigma \in X(k-1)} \deg(\sigma)$.

(k, d, ϵ)-Expanders

A family of Complexes $\{X_n\}_n$ with $f_0(X_n) \to \infty$ such that

$$D_{k-1}(X_n) \leq d \quad \text{and} \quad h_{k-1}(X_n) \geq \epsilon.$$

Random Complexes as Expanders

$Y \in Y_k(n, p = \frac{k^2 \log n}{n})$ is a.a.s. a $(k, \log n, 1)$-expander.

Problem

Do there exist (k, d, ϵ)-expanders with $k \geq 2$ and fixed d, ϵ ?

Latin Squares

Definitions

$\mathbb{S}_n$ = Symmetric group on $[n]$.
$(\pi_1, \ldots, \pi_k) \in \mathbb{S}_n^k$ is legal if $\pi_i(\ell) \neq \pi_j(\ell)$ for all ℓ and $i \neq j$.
A Latin Square is a legal n-tuple $L = (\pi_1, \ldots, \pi_n) \in \mathbb{S}_n^n$.
$\mathcal{L}_n$ = Latin squares of order n with uniform measure.

The Usual Picture

$L = (\pi_1, \ldots, \pi_n) \leftrightarrow T_L \in M_{n \times n}([n])$
$T_L(i, \pi_k(i)) = k$ for $1 \leq i, k \leq n$.

Example for $n = 4$

$\pi = (1234)$

$L = (Id, \pi, \pi^2, \pi^3)$

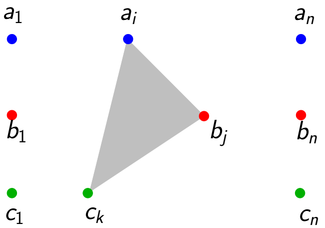
$T_L =$

1	2	3	4
4	1	2	3
3	4	1	2
2	3	4	1

The Complete 3-Partite Complex

$$V_1 = \{a_1, \ldots, a_n\}\ ,\ V_2 = \{b_1, \ldots, b_n\}\ ,\ V_3 = \{c_1, \ldots, c_n\}$$

$$T_n = V_1 * V_2 * V_3 = \{\sigma \subset V : |\sigma \cap V_i| \leq 1 \text{ for } 1 \leq i \leq 3\}$$



$$T_n \simeq S^2 \vee \cdots \vee S^2 \qquad (n-1)^3 \text{ times}$$

Latin Square Complexes

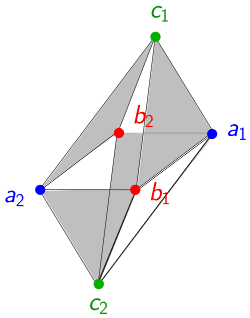
$L = (\pi_1, \ldots, \pi_n) \in \mathcal{L}_n$ defines a complex $Y(L) \subset T_n$ by

$$Y(L)(2) = \left\{ \, [a_i, b_j, c_{\pi_i(j)}] \;\; : \;\; 1 \leq i, j \leq n \right\}.$$

Example: $n = 2$

$$L = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 1 \\ \hline \end{array}$$

$Y(L) =$



$$Y\left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 1 \\ \hline \end{array} \right) \cup Y\left(\begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 & 2 \\ \hline \end{array} \right) = T_2$$

Random Latin Squares Complexes

Multiple Latin Squares

For $\underline{L}^d = (L_1, \ldots, L_d) \in \mathcal{L}_n^d$ let $Y(\underline{L}^d) = \cup_{i=1}^d Y(L_i)$.

The Probability Space $\mathcal{Y}(n, d)$

$\mathcal{L}_n^d$ = d-tuples of Latin squares of order n with uniform measure.
$\mathcal{Y}(n, d) = \{Y(\underline{L}^d) : \underline{L}^d \in \mathcal{L}_n^d\}$ with induced measure from $\mathcal{L}_n^d$.

Theorem [Lubotzky-M]:

There exist $\epsilon > 0, d < \infty$ such that

$$\lim_{n \to \infty} \Pr\left[Y \in \mathcal{Y}(n, d) : h_1(Y) > \epsilon\right] = 1.$$

Remark: $\epsilon = 10^{-11}$ and $d = 10^{11}$ will do.

Idea of Proof

Fix $0 < c < 1$ and let $\phi \in C^1(T_n; \mathbb{F}_2)$.

$$\phi \text{ is } \begin{cases} c - \text{small} & \text{if } \|[\phi]\| \leq cn^2 \\ c - \text{large} & \text{if } \|[\phi]\| \geq cn^2 \end{cases}$$

c-Small Cochains

Lower bound on expansion in terms of the spectral gap of the vertex links.

c-Large Cochains

Expansion is obtained by means of a new large deviations bound for the probability space $\mathcal{L}_n$ of Latin squares.

2-Expansion and Spectral Gap

Notation

For a complex $T_n^{(1)} \subset Y \subset T_n$ let:

$Y_v = \mathrm{lk}(Y, v) =$ the link of $v \in V$.

$\lambda_v =$ spectral gap of the $n \times n$ bipartite graph Y_v.

$\tilde{\lambda} = \min_{v \in V} \lambda_v$.

$d = D_1(Y) =$ maximum edge degree in Y.

Theorem [LM]:

If $\|[\phi]\| \leq cn^2$ then

$$\|d_1\phi\| \geq \left(\frac{(1 - c^{1/3})\tilde{\lambda}}{2} - \frac{d}{3} \right) \|[\phi]\|.$$

Spectral Gap of Random Graphs

Random Bipartite Graphs

$\tilde{\pi} = (\pi_1, \ldots, \pi_d) \in \mathbb{S}_n^d$ defines a graph $G = G(\tilde{\pi})$ by

$$E(G) = \{\ (i, \pi_j(i))\ :\ 1 \leq i \leq n, 1 \leq j \leq d\ \} \subset [n]^2.$$

$\mathcal{G}(n, d)$= uniform probability space $\{G(\tilde{\pi}) : \tilde{\pi} \in \mathbb{S}_n^d\}$.

Theorem [Friedman]:

For a fixed $d \geq 100$:

$$\Pr[G \in \mathcal{G}(n, d)\ :\ \lambda_2(G) > d - 3\sqrt{d}] = 1 - O(n^{-2}).$$

Expansion of c-Small Cochains

Links as Random Graphs

Let $Y = Y(\underline{L}^d)$ be a random complex in $\mathcal{Y}(n,d)$.
Then $Y_v = \mathrm{lk}(Y,v)$ is a random graph in $\mathcal{G}(n,d)$.
Therefore

$$\Pr[\ \tilde{\lambda} \geq d - 3\sqrt{d}\ ] = 1 - O(n^{-1}).$$

Corollary:

Let $d \geq 100$ and $c < 10^{-3}$. If $\|[\phi]\| \leq cn^2$ then

$$\frac{\|d_1\phi\|}{\|[\phi]\|} \geq \frac{(1-c^{1/3})\tilde{\lambda}}{2} - \frac{d}{3}$$

$$\geq \frac{(1-c^{1/3})(d-3\sqrt{d})}{2} - \frac{d}{3} > 1.$$

Large Deviations for Latin Squares

The Random Variable $f_{\mathcal{E}}$

$\mathcal{E}$ - a family of 2-simplices of T_n, $|\mathcal{E}| \geq cn^3$.
For a Latin square $L \in \mathcal{L}_n$ let

$$f_{\mathcal{E}}(L) = |Y(L) \cap \mathcal{E}|.$$

Then

$$E[f_{\mathcal{E}}] = \frac{|\mathcal{E}|}{n} \geq cn^2.$$

Theorem [LM]:

For all $n \geq n_0(c)$

$$\Pr[L \in \mathcal{L}_n \ : \ f_{\mathcal{E}}(L) < 10^{-3}c^2n^2] < e^{-10^{-3}c^2n^2}.$$

Homological Connectivity of $\mathcal{Y}(n,d)$

Theorem [LM]:

$$Y \in \mathcal{Y}(n, 10^{11}) \quad \Rightarrow \quad H_1(Y;\mathbb{F}_2) = 0 \quad \text{a.a.s.}$$
$$Y \in \mathcal{Y}(n, 10^{4}) \quad \Rightarrow \quad H_1(Y;\mathbb{R}) = 0 \quad \text{a.a.s.}$$

Claim:

$$\lim_{n\to\infty} \Pr\left[Y \in \mathcal{Y}(n,3) : H_1(Y;\mathbb{F}_2) \neq 0\right] \geq 1 - \frac{17e^{-3}}{2} \doteq 0.57.$$

Conjecture:

$$Y \in \mathcal{Y}(n, 4) \quad \Rightarrow \quad H_1(Y;\mathbb{F}_2) = 0 \quad a.a.s.$$

More Open Problems

- Find explicit constructions of bounded degree expanders.
- Are some (most) Ramanujan complexes high dimensional expanders?
- The model $\mathcal{Y}(n, d)$ generalizes to higher dimensions. Does the theorem remain true there?

Higher Laplacians

A positive weight function $c(\sigma)$ on the simplices of X induces an Inner product on $C^k(X) = C^k(X;\mathbb{R})$:

$$(\phi,\psi) = \sum_{\sigma\in X(k)} c(\sigma)\phi(\sigma)\psi(\sigma) \quad .$$

Adjoint $\quad d_k^* : C^{k+1}(X) \to C^k(X)$

$$(d_k\phi,\psi) = (\phi, d_k^*\psi) \ .$$

$$C^{k-1}(X) \underset{d_{k-1}^*}{\overset{d_{k-1}}{\rightleftarrows}} C^k(X) \underset{d_k^*}{\overset{d_k}{\rightleftarrows}} C^{k+1}(X)$$

The reduced k-Laplacian of X is the positive semidefinite operator

$$\Delta_k = d_{k-1}d_{k-1}^* + d_k^* d_k : C^k(X) \to C^k(X) \quad .$$

Matrix Representation of Δ_k

For the constant weight function $c \equiv 1$, the matrix form of the Laplacian is

$$\Delta_k(\sigma,\tau) = \begin{cases} \deg(\sigma) + k + 1 & \sigma = \tau \\ (\sigma : \sigma \cap \tau) \cdot (\tau : \sigma \cap \tau) & |\sigma \cap \tau| = k \ , \ \sigma \cup \tau \notin X \end{cases}$$

Relation with the Graph Laplacian

Let G = 1-skeleton of X

$$\Delta_0 = L_G + J$$

$$\mu_0(X) = \lambda_2(G)$$

Harmonic Cochains

The space of Harmonic k-cochains

$$\ker \Delta_k = \{\phi \in C^k(X) : d_k\phi = 0 \ , \ d_{k-1}^*\phi = 0\}.$$

Simplicial Hodge Theorem:

$$C^k(X) = \mathrm{Im}\, d_{k-1} \oplus \ker \Delta_k \oplus \mathrm{Im}\, d_k^* \ .$$

$$\ker \Delta_k \cong \tilde{\mathrm{H}}^k(X;\mathbb{R}).$$

$\mu_k(X) =$ minimal eigenvalue of Δ_k.

A Vanishing Criterion:

$$\mu_k(X) > 0 \Leftrightarrow \tilde{\mathrm{H}}_k(X;\mathbb{R}) = 0.$$

Spectral Gap and Colorful Simplices

$\Delta_{n-1}^{(k-1)} \subset X \subset \Delta_{n-1}^{(k)}$ with vertex coloring: $[n] = V_0 \cup \cdots \cup V_k$.
Number of colorful k-simplices:

$$e(V_0, \ldots, V_k) = |\{\sigma \in X(k) : |\sigma \cap V_i| = 1 \;\; \forall\, 0 \leq i \leq k\}|.$$

Theorem [Parzanchevski-Rosenthal-Tessler]:

Let c be the constant weight function $c(\sigma) \equiv 1$. Then

$$e(V_0, \ldots, V_k) \geq \frac{\prod_{i=0}^{k} |V_i|}{n} \cdot \mu_{k-1}(X).$$

Sketch of Proof

Define $\psi \in C^k(\Delta_{n-1})$ by

$$\psi([v_0, \ldots, v_k]) = \begin{cases} sgn(\pi) & v_{\pi(i)} \in V_i \ \ \forall\, 0 \leq i \leq k \\ 0 & [v_0, \ldots, v_k] \text{ is not colorful.} \end{cases}$$

Let $\phi = d^*_{k-1}\psi \in C^{k-1}(\Delta_{n-1}) = C^{k-1}(X)$. Then:

$$(\Delta_{k-1}\phi, \phi) = (d_{k-1}\phi, d_{k-1}\phi) = n^2 \cdot e(V_0, \ldots, V_k)$$

$$(\phi, \phi) = n \prod_{i=0}^{k} |V_i|.$$

Therefore, by the variational principle:

$$\mu_{k-1}(X) \leq \frac{(\Delta_{k-1}\phi, \phi)}{(\phi, \phi)} = \frac{n \cdot e(V_0, \ldots, V_k)}{\prod_{i=0}^{k} |V_i|}.$$

Eigenvalues and Cohomology

Let X be a pure d-dimensional complex with weight function:

$$c(\sigma) = (d - \dim \sigma)! |\{\tau \in X(d) : \tau \supset \sigma\}|.$$

For $\tau \in X$ consider the link $X_\tau = \mathrm{lk}(X, \tau)$ with a weight function given by $c_\tau(\alpha) = c(\tau\alpha)$.

Theorem [Garland '72]:

Let $0 \leq \ell < k < d$. Then:

$$\min_{\tau \in X(\ell)} \mu_{k-\ell-1}(X_\tau) > \frac{\ell + 1}{k + 1} \quad \Rightarrow \quad H^k(X; \mathbb{R}) = 0.$$

In particular:

$$\min_{\tau \in X(d-2)} \mu_0(X_\tau) > \frac{d - 1}{d} \quad \Rightarrow \quad H^{d-1}(X; \mathbb{R}) = 0.$$

Sketch of Proof I

For $\phi \in C^k(X)$ define $\phi_\tau \in C^{k-\ell-1}(X_\tau)$ by $\phi_\tau(\alpha) = \phi(\tau\alpha)$.

Garland's Identity

$$\binom{k}{\ell+1}(\Delta_k \phi, \phi) = \sum_{\tau \in X(\ell)} (\Delta_{k-\ell-1}\phi_\tau, \phi_\tau) - \binom{k}{\ell}\|\phi\|^2.$$

Proof of Garland's Theorem:

Suppose that

$$\min_{\tau \in X(\ell)} \mu_{k-\ell-1}(X_\tau) > \frac{\ell+1}{k+1}$$

and let $0 \neq \phi \in C^k(X)$ such that $\Delta_k \phi = \mu_k(X)\phi$.

Sketch of Proof II

By Garland's identity:

$$
\begin{aligned}
\mu_k(X)\binom{k}{\ell+1}\|\phi\|^2 &= \binom{k}{\ell+1}(\Delta_k\phi,\phi) \\
&= \sum_{\tau\in X(\ell)} (\Delta_{k-\ell-1}\phi_\tau,\phi_\tau) - \binom{k}{\ell}\|\phi\|^2 \\
&\geq \min_{\tau\in X(\ell)} \mu_{k-\ell-1}(X_\tau) \sum_{\tau\in X(\ell)} \|\phi_\tau\|^2 - \binom{k}{\ell}\|\phi\|^2 \\
&\geq \left(\min_{\tau\in X(\ell)} \mu_{k-\ell-1}(X_\tau)\binom{k+1}{\ell+1} - \binom{k}{\ell}\right)\|\phi\|^2 \\
&> \left(\frac{\ell+1}{k+1}\binom{k+1}{\ell+1} - \binom{k}{\ell}\right)\|\phi\|^2 = 0.
\end{aligned}
$$

Complexes with Expanding Links

The Projective Plane Graph

$G_q = (V_q, E_q)$: points vs. lines graph of $PG(2,q)$.

$$|V_q| = 2(q^2+q+1) \quad , \quad |E_q| = (q+1)(q^2+q+1).$$

Spectral Gap: $\mu_0(G_q) = 1 - \frac{\sqrt{q}}{q+1}$.

If $q \geq d^2$ then $\mu_0(G_q) > \frac{d-1}{d}$. This implies the following

Theorem [Garland]:

Let $q \geq d^2$ and let X be a pure d-dimensional complex such that $\mathrm{lk}(X,\tau) \cong G_q$ for all $\tau \in X(d-2)$.
Then $H_{d-1}(X;\mathbb{R}) = 0$.

Cohomology of Discrete Subgroups

$\mathbb{K}$ a local field with residue field $\mathbb{F}_q$.
Γ a torsion-free discrete cocompact subgroup of $SL_{d+1}(\mathbb{K})$.

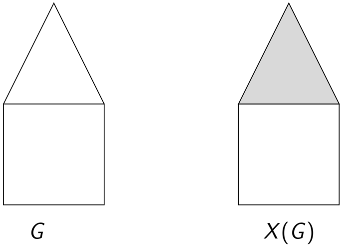
Theorem [Garland]:
If $q \geq d^2$ then $H^i(\Gamma;\mathbb{R}) = 0$ for $0 < i < d$.

Sketch of Proof:
$\mathcal{B} = \tilde{A}_d(\mathbb{K})$ - the affine building associated to $SL_{d+1}(\mathbb{K})$.
$\mathcal{B}$ is a contractible complex with a free Γ action.
The quotient space $\mathrm{B}\Gamma = \mathcal{B}/\Gamma$ is a pure d-dimensional complex such that $\mathrm{lk}(\mathrm{B}\Gamma, \tau) \cong G_q$ for all $\tau \in \mathrm{B}\Gamma(d-2)$.
Therefore for all $0 < i < d$

$$H^i(\Gamma;\mathbb{R}) = H^i(\mathrm{B}\Gamma;\mathbb{R}) = 0.$$

Flag Complexes

The flag complex $X(G)$ of a graph $G = (V, E)$:
Vertex set: V, Simplices: all cliques σ of G.



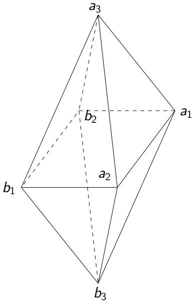
Remark:

The first subdivision of a complex is a flag complex.

Face Numbers of Flag Complexes

Octahedral n-Sphere

$(S^0)^{*(k+1)} =$
$\{a_1, b_1\} * \cdots * \{a_{k+1}, b_{k+1}\}$



Proposition [M '03]:

If $\tilde{\mathrm{H}}_k(X(G)) \neq 0$ then for all j:

$$f_j(X(G)) \geq f_j((S^0)^{*(k+1)}) = \binom{k+1}{j+1} 2^{j+1}.$$

Homology of Flag Complexes of Random Graphs

Let $\epsilon > 0$ be fixed and let $G \in G(n, p)$.

Theorem [Kahle '12]:

$$p \leq n^{-\frac{1}{k}-\epsilon} \quad \Rightarrow \quad H_k(X(G); \mathbb{Z}) = 0 \ \text{ a.a.s.}$$

$$p \geq \left(\frac{(\frac{k}{2} + 1 + \epsilon) \log n}{n} \right)^{\frac{1}{k+1}} \quad \Rightarrow \quad H_k(X(G); \mathbb{R}) = 0 \ \text{ a.a.s.}$$

Theorem [DeMarco-Hamm-Kahn '12]:

$$p \geq \left(\frac{(\frac{3}{2} + \epsilon) \log n}{n} \right)^{\frac{1}{2}} \quad \Rightarrow \quad H_1(X(G); \mathbb{F}_2) = 0 \ \text{ a.a.s.}$$

Vanishing of $H_k(X(G); \mathbb{R})$

Let $C_k = \frac{k}{2} + 1 + \epsilon$ and let $p = \left(\frac{C_k \log n}{n}\right)^{\frac{1}{k+1}}$.

Claim 1: The $(k+1)$-skeleton $Y = X(G)^{(k+1)}$ is a.a.s. pure:

$$
\begin{aligned}
&E[\#\sigma \in Y(k) \text{ such that } \sigma \not\subset (k+1) - \text{face of } Y] \\
&= \binom{n}{k+1} p^{\binom{k+1}{2}} (1 - p^{k+1})^{n-k-1} \\
&\leq n^{k+1} \left(\frac{C_k \log n}{n}\right)^{\frac{k}{2}} \left(1 - \frac{C_k \log n}{n}\right)^{n-k-1} = O(n^{-\frac{\epsilon}{2}}).
\end{aligned}
$$

Claim 2: $\mu_0(\mathrm{lk}(Y, \tau)) = 1 - o(1)$ a.a.s. for all $\tau \in Y(k-1)$.

Claims 1 & 2 + Garland's Thm. $\Rightarrow$ $H_k(X(G); \mathbb{R}) = 0$ a.a.s.

Eigenvalues of Flag Complexes

$G = (V, E)$ graph, $|V| = n$, $X = X(G)$ with weights $c(\sigma) \equiv 1$.
$\mu_k = \mu_k(X) =$ minimal eigenvalue of Δ_k on X.

Theorem [Aharoni-Berger-M]:

For $k \geq 1$

$$k\mu_k \geq (k+1)\mu_{k-1} - n.$$

In particular:

$$\mu_k \geq (k+1)\lambda_2 - kn.$$

Corollary:

$$\lambda_2(G) > \frac{kn}{k+1} \;\Rightarrow\; \mu_k > 0 \;\Rightarrow\; \tilde{\mathrm{H}}^k(X(G)) = 0.$$

Example: Turán Graph

$|V_1| = \cdots = |V_k| = \ell$, $n = k\ell$, $m = (\ell - 1)^k$.
$T_k(n)$ - the complete k-partite graph on $V_1 \cup \cdots \cup V_k$.

Spectral gap

$$\lambda_2(T_k(n)) = \frac{(k-1)n}{k}.$$

Flag complex

$$X(T_k(n)) = V_1 * \cdots * V_k \simeq \bigvee_{i=1}^{m} S^{k-1}.$$

$$\dim \tilde{H}_{k-1}(X(T_k(n)); \mathbb{R}) = m \neq 0.$$

Preliminaries

For $\tau \in X(k-1)$ let $\deg(\tau) = |\{\sigma \in X(k) \ : \ \sigma \supset \tau\}|$

Easy Fact: For $\sigma \in X(k)$

$$\sum_{\tau \in \sigma(k-1)} \deg(\tau) - k \deg(\sigma) \leq n \ .$$

For $\phi \in C^k(X)$ and a vertex $u \in V$ define $\phi_u \in C^{k-1}(X)$ by

$$\phi_u(\tau) = \begin{cases} \phi(u\tau) & u \in \mathrm{lk}(\tau) \\ 0 & \text{otherwise} \end{cases}$$

By double counting $\sum_{u \in V} ||\phi_u||^2 = (k+1)||\phi||^2$.

Sketch of Proof

Key Identity:

$$k(\Delta_k \phi, \phi) =$$

$$\sum_{u \in V} (\Delta_{k-1}\phi_u, \phi_u) - \sum_{\sigma \in X(k)} \Big(\sum_{\tau \in \sigma(k-1)} \deg(\tau) - k \deg(\sigma) \Big) \phi(\sigma)^2 \quad .$$

Choose an eigenvector $0 \neq \phi \in C^k(X)$ with $\Delta_k \phi = \mu_k \phi$. Then

$$k\mu_k ||\phi||^2 \geq \sum_{u \in V} (\Delta_{k-1}\phi_u, \phi_u) - n \sum_{\sigma \in X(k)} \phi(\sigma)^2 \quad \geq$$

$$\mu_{k-1} \sum_{u \in V} ||\phi_u||^2 - n||\phi||^2 = ((k+1)\mu_{k-1} - n)||\phi||^2 \quad .$$

Eigenvalues and Connectivity of $\mathrm{I}(G)$

The independence complex $\mathrm{I}(G)$

Vertex set: V, Simplices: all independent sets σ of G.

Homological connectivity

$$\eta(Y) = 1 + \min\{i : \tilde{H}_i(Y) \neq 0\}.$$

Theorem [ABM]:

For a graph G on n vertices

$$\eta(\mathrm{I}(G)) \geq \frac{n}{\lambda_n(G)}.$$

Bipartite Matching

$A_1, \ldots, A_m$ finite sets.
A System of Distinct Representatives (SDR):
a choice of distinct $x_1 \in A_1, \ldots, x_m \in A_m$.

A_1	A_2	A_3
1	1	
		2
3	3	3

$\exists$ SDR

A_1	A_2	A_3
1	1	
2		2

$\nexists$ SDR

Hall's Theorem (1935)

$(A_1, \ldots, A_m)$ has an SDR iff
$|\cup_{i \in \mathrm{I}} A_i| \geq |\mathrm{I}|$ for all $\mathrm{I} \subset [m] = \{1, \ldots, m\}$.

Hypergraph Matching

A Hypergraph is a family of sets $\mathcal{F} \subset 2^V$
$(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ a sequence of m hypergraphs
A System of Disjoint Representatives (SDR) for $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ is a choice of pairwise disjoint $F_1 \in \mathcal{F}_1, \ldots, F_m \in \mathcal{F}_m$

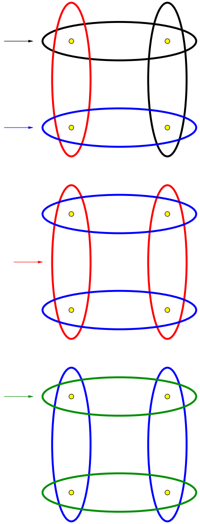
When do $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ have an SDR?

The problem is NP-Complete even if all $\mathcal{F}_i$'s consist of 2-element sets. Therefore, we cannot expect a "good" characterization as in Hall's Theorem.

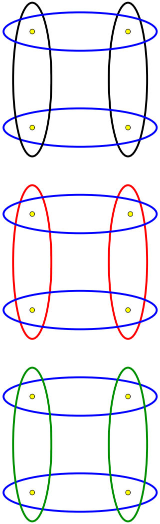
There are however some interesting sufficient conditions ...

Do $(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4)$ have an SDR?

$\exists$ SDR



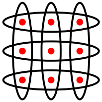
$\nexists$ SDR



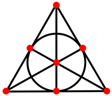
The Aharoni-Haxell Theorem

A Matching is a hypergraph $\mathcal{M}$ of pairwise disjoint sets.
The Matching Number $\nu(\mathcal{F})$ of a hypergraph $\mathcal{F}$ is the maximal size $|\mathcal{M}|$ of a matching $\mathcal{M} \subset \mathcal{F}$.

$\nu(\mathcal{F}) = 3$



$\nu(\mathcal{F}) = 1$



The Aharoni-Haxell Theorem
$\mathcal{F}_1, \ldots, \mathcal{F}_m \subset \binom{V}{r}$ such that for all $\mathrm{I} \subset [m]$

$$\nu\Big(\bigcup_{i \in \mathrm{I}} \mathcal{F}_i\Big) > r(|\mathrm{I}| - 1) \quad .$$

Then $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ has an SDR.

A Fractional Extension

A Fractional Matching of a hypergraph $\mathcal{F}$ on V is a function $f : \mathcal{F} \to \mathbb{R}_+$ such that $\sum_{F \ni v} f(F) \leq 1$ for all $v \in V$.
The Fractional Matching Number $\nu^*(\mathcal{F})$ is $\max_f \sum_{F \in \mathcal{F}} f(F)$ over all fractional matchings f .

Example: The Finite Projective Plane $\mathcal{P}_n$

$\nu(\mathcal{P}_n) = 1 \quad , \quad \nu^*(\mathcal{P}_n) = \frac{n^2+n+1}{n+1}$

Theorem [Aharoni-Berger-M]:

$\mathcal{F}_1, \ldots, \mathcal{F}_m \subset \binom{V}{r}$ such that for all $\mathrm{I} \subset [m]$

$$\nu^*(\bigcup_{i \in \mathrm{I}} \mathcal{F}_i) > r(|\mathrm{I}| - 1) \quad .$$

Then $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ has an SDR.