3-monotone interpolation

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Motivation

How to test whether a given planar point set with distinct x-coordinates lies on the graph of a

1. Non-negative function? (i.e. (-2)th derivative convex)
2. Non-decreasing function? (i.e. (-1)th derivative convex)
3. Convex function? (i.e. 0th derivative convex)
4. 3-monotone function (i.e. 1st derivative convex)?

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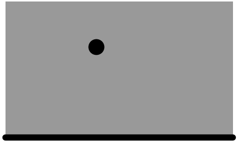
- For every point, take the zero-degree polynomial passing through it. Its leading coefficient has to be non-negative.



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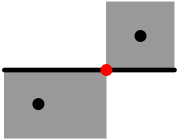
- For every pair of points, take the degree 1 polynomial through them. The leading coefficient has to be non-negative.



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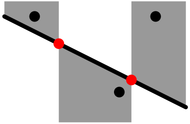
- For every triple of points, take the degree 2 polynomial through them. The leading coefficient has to be non-negative.



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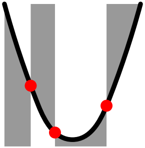
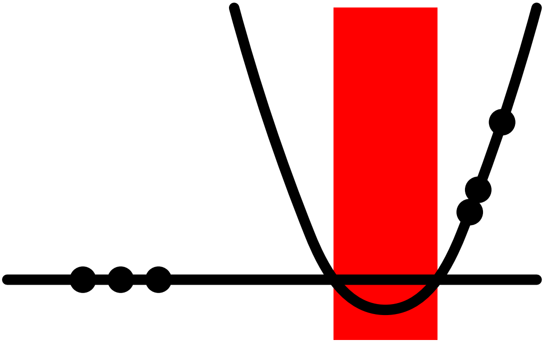


We don't know any effective characterization!

4 points are not enough: Rote's counterexample

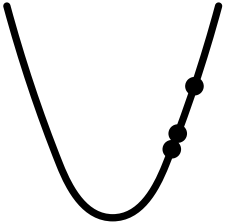


k points are not enough

Theorem (Cibulka, Matoušek, P; '14)

Given any natural number k, one cannot decide whether a planar point set admits a 3-monotone interpolation by checking k-tuples of points only.

Constructing counterexample



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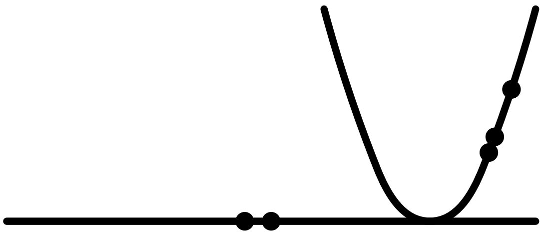

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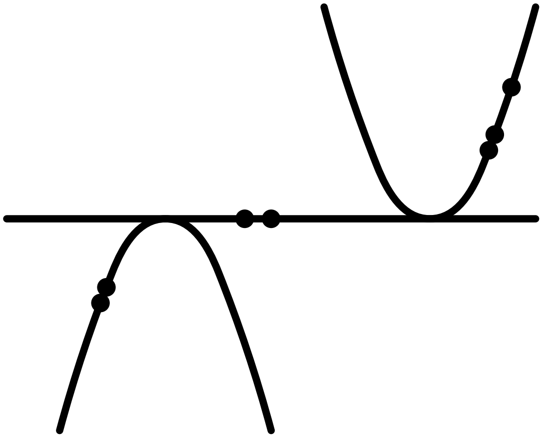

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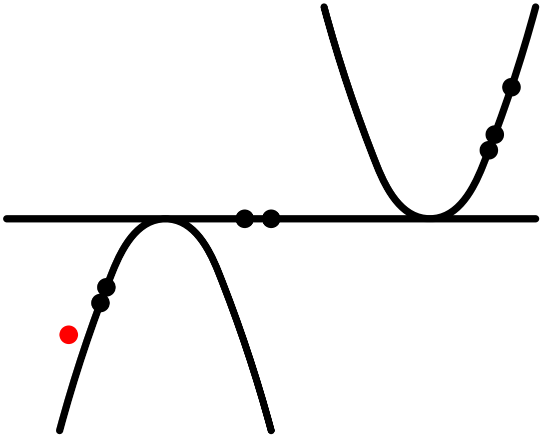

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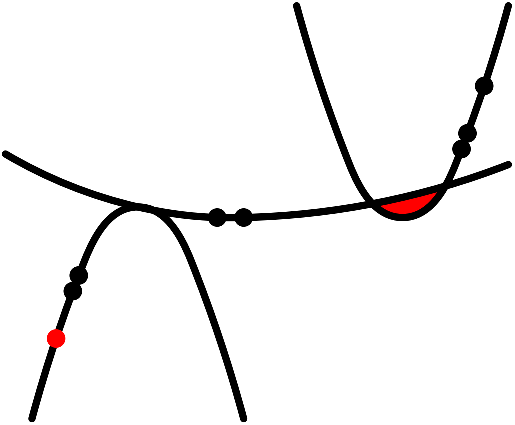

Ramsey type result

Let P be a planar point set with distinct x-coordinates.

Theorem (Erdős-Szekeres, 1935)

If $|P| > (r-1)\cdot(s-1)$, then P contains r points on the graph of a non-decreasing function, or s on the graph of a non-increasing one.

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Example

If $|P| > (r-1)+(s-1)$ then P contains r points on the graph of a non-negative function, or s on the graph of a non-positive one.

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Theorem (Cibulka, Matoušek, P; '14)

If $|P| > f(r,s)$, then P contains r points on the graph of a 3-monotone function, or s on the graph of a 3-antimonotone one.

B-splines

Definition (B-spline)

Let $a < b < c < d$ be distinct real numbers. The **B**-spline determined by a, b, c, d is the piecewise quadratic function $u_{a,b,c,d} : \mathbb{R} \to \mathbb{R}$ defined by

$$u_{a,b,c,d}(x) := \begin{cases} 0 & x \in [-\infty, a] \\ \frac{(x-a)^2}{(c-a)(b-a)} & x \in [a, b] \\ \frac{(x-a)(c-x)}{(c-a)(c-b)} + \frac{(d-x)(x-b)}{(d-b)(c-b)} & x \in [b, c] \\ \frac{(d-x)^2}{(d-b)(d-c)} & x \in [c, d] \\ 0 & x \in [d, \infty]. \end{cases}$$

Sketch of the proof

Proof.

1. For every fourtuple F of points consider the cubic polynomial passing through all of them.
2. Choose a sufficiently large set with all leading coefficients ℓ_F non-negative / non-positive (for antimonotone)
3. Define a fivetuple $p_1, \ldots, p_5$ to be v-positive, if

$$\frac{\ell_{p_1,\ldots,p_4}}{u_{x_1,\ldots,x_4}(x_3)} - \frac{\ell_{p_2,\ldots,p_5}}{u_{x_2,\ldots,x_4}(x_3)} > 0$$

4. Pick a subset with all fivetuples of the same sign:

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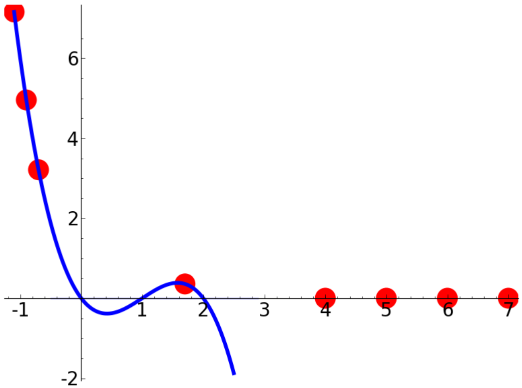

Thank you for your questions!