# Recent Progress on Hill's Conjecture

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Czech Republic

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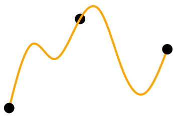

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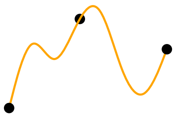

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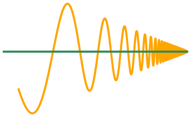

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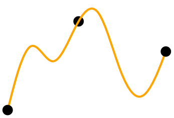

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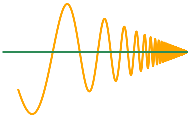

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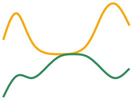

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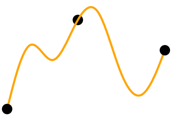

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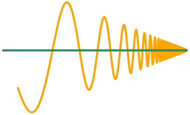

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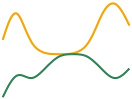

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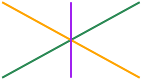

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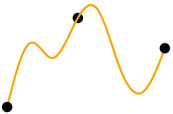

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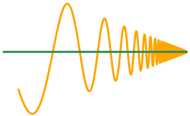

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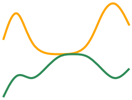

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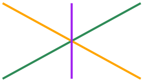

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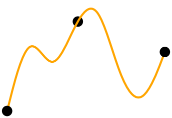

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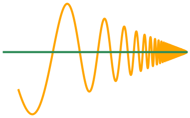

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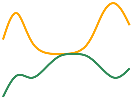

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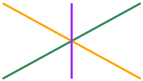

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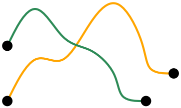

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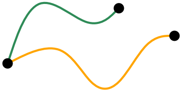

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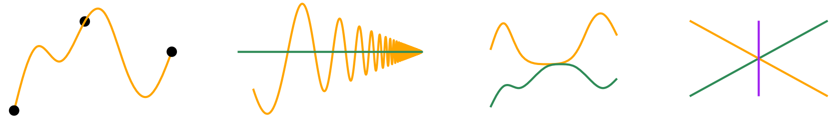

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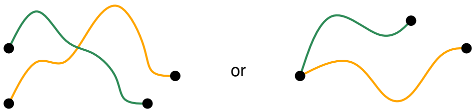


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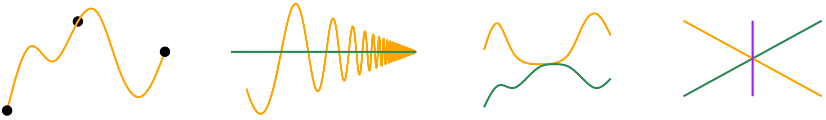

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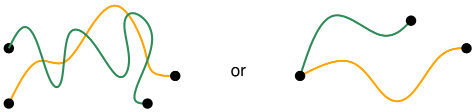


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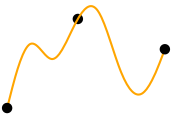
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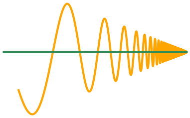
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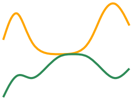
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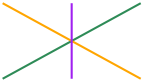
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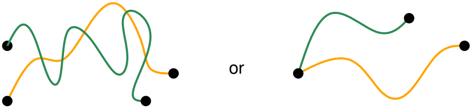


- In a semisimple drawing independent edges may cross more than once.
- A drawing is called $x$-monotone if edges are $x$-monotone curves.

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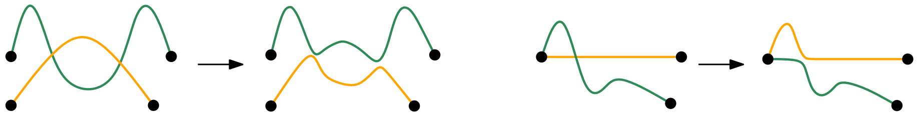

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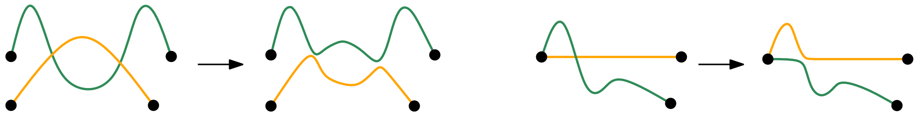

- The monotone crossing number $\text{mon-cr}(G)$ of $G$ is the minimum number of crossings $\text{cr}(D)$ in $D$ taken over all <u>$x$-monotone</u> drawings $D$ of $G$.

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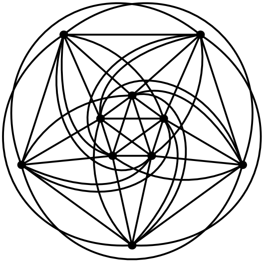

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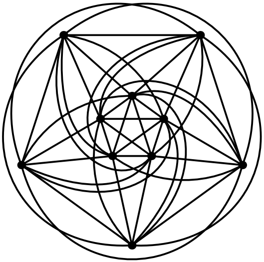

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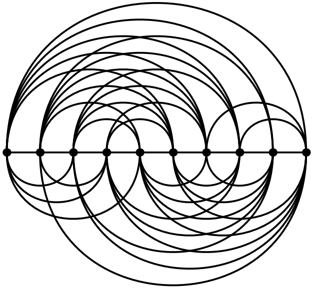

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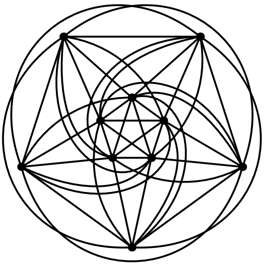

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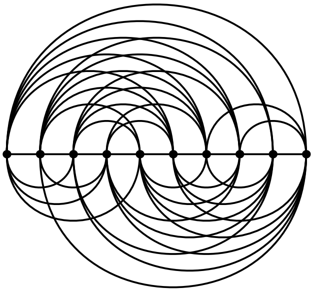

- A drawing is 2-page if the vertices are placed on a line $\ell$ and each edge is fully contained in a halfspace determined by $\ell$.

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  - weakly semisimple *s*-shellable drawings.
- Since 2-page drawings are *x*-monotone, we have $\text{mon-cr}(K_n) \leq Z(n)$.

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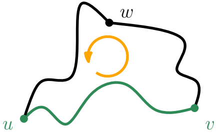


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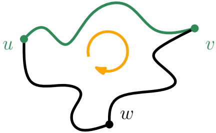


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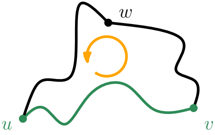


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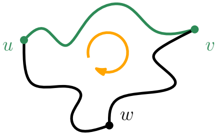


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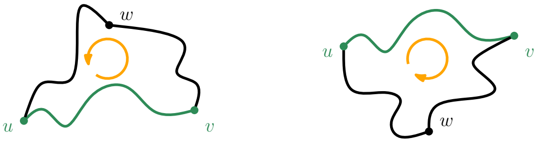


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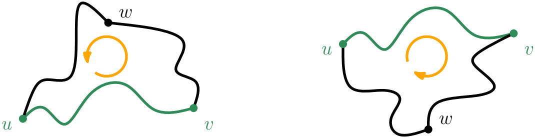


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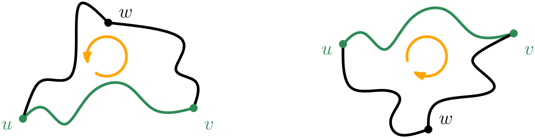


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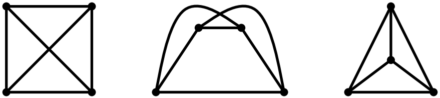

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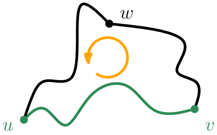


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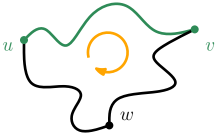


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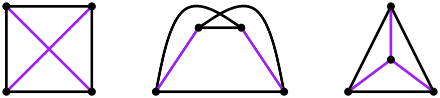

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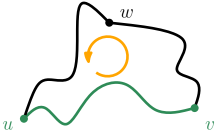


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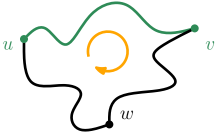


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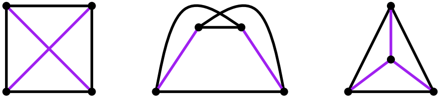

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**Lemma**

For a simple drawing $D$ of $K_n$ we get $\mathrm{cr}(D) = 3\binom{n}{4} - \sum_{k=0}^{\lfloor n/2 \rfloor - 1} k(n-2-k)E_k(D)$.

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- That is, we want a lower bound for $E_{\leq\leq k}(D)$.

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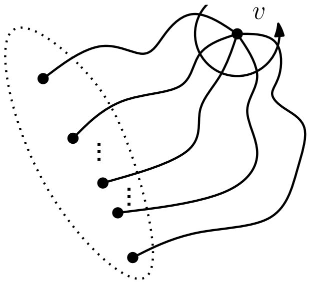


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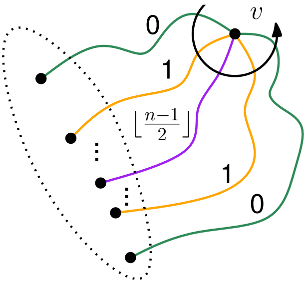


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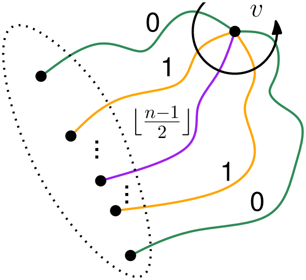


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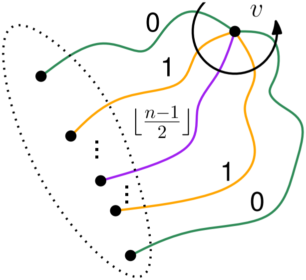


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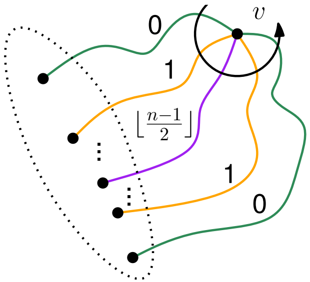


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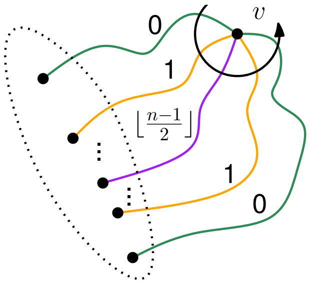


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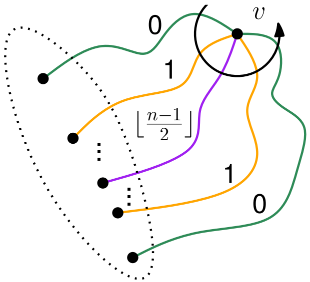


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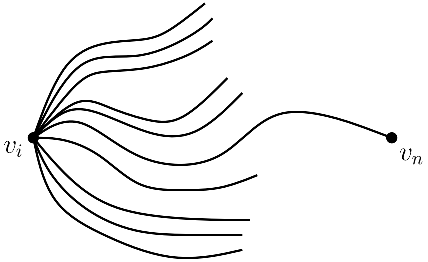


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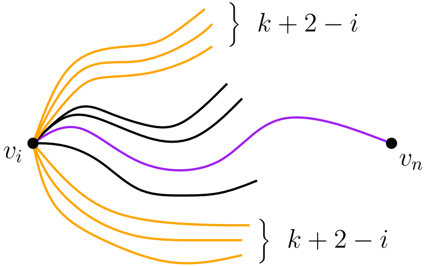


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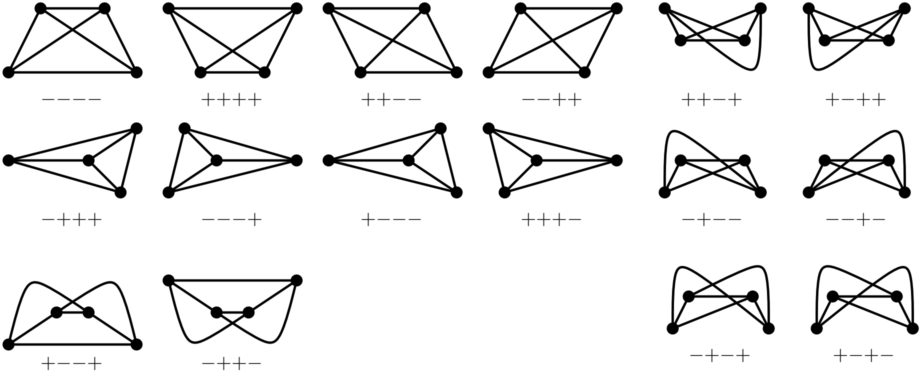

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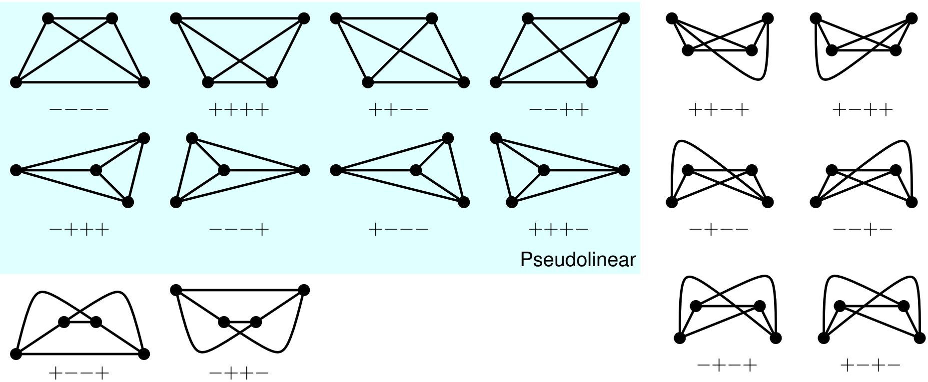

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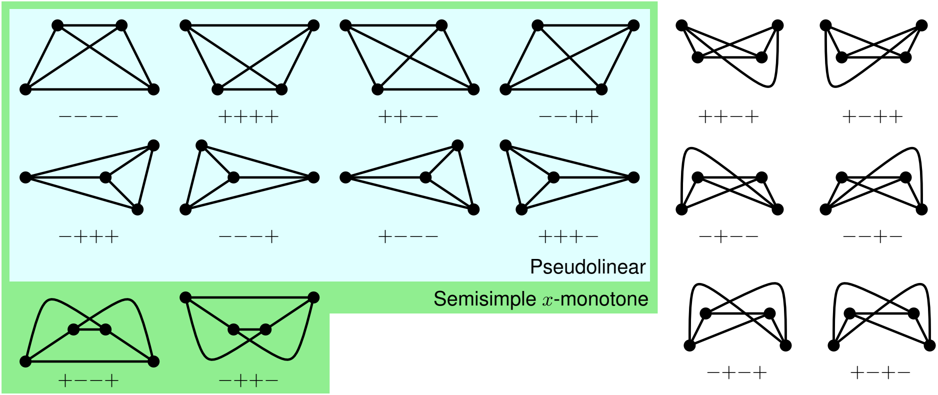

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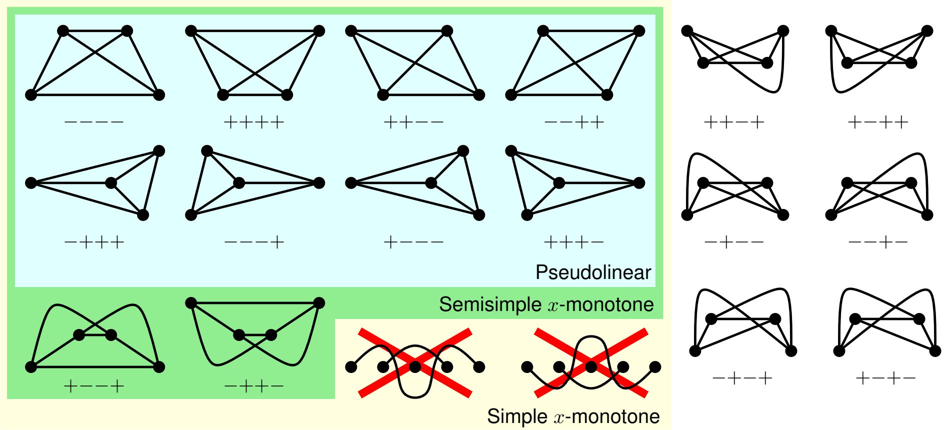

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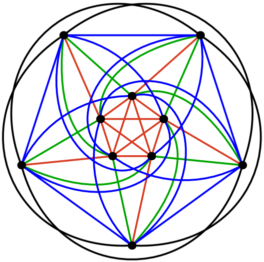

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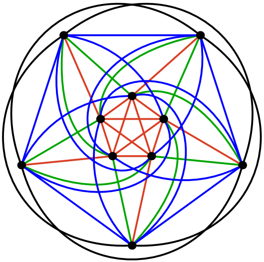

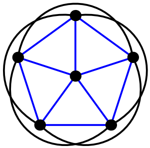

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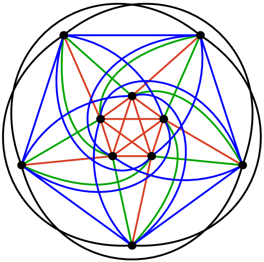

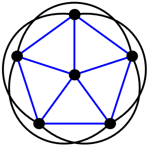

- Here we have $E_0 = 5$ and $E_1 = 0$, hence $E_{\leq\leq 1} = 10 < 12 = 3\binom{1+3}{3}$.

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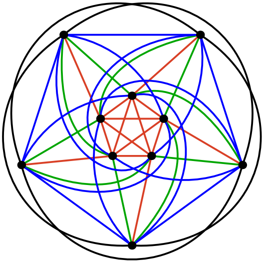

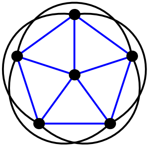

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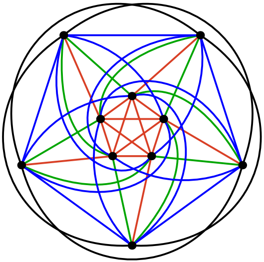
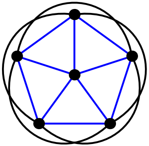
## General Drawings

- There are (optimal) drawings of $K_n$ where $E_{\leq\leq k}(D) \geq 3\binom{k+3}{3}$ does not hold.





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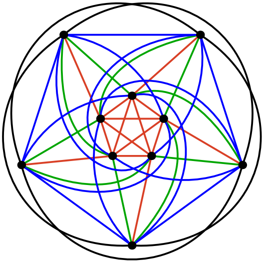

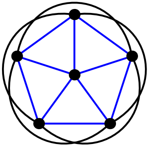

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- Implies Hill's conjecture. All drawings we have found satisfy this conjecture.