On the Hamiltonicity and related properties of variants of the Delaunay triangulation

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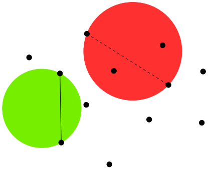
Empty circle property



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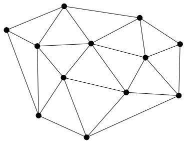

Some good properties

- If not all points are collinear and no four points define an empty circle, then $\mathrm{DG}(S)$ is a triangulation (maximal plane graph).

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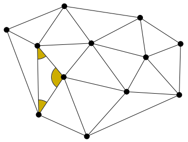

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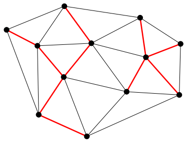

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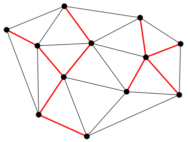

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- What about more involved properties?

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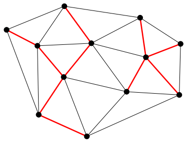
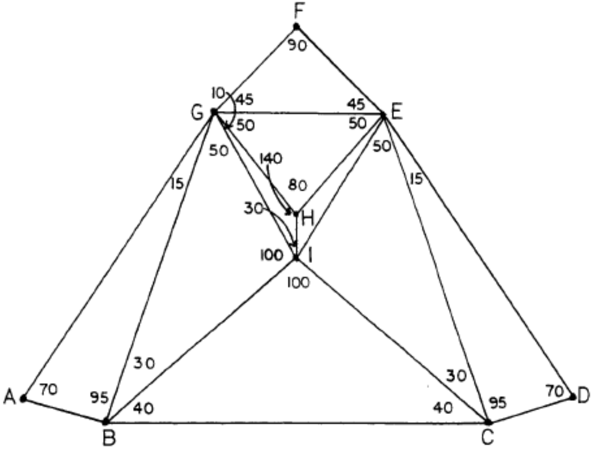


Some good properties

- If not all points are collinear and no four points define an empty circle, then DG(S) is a triangulation (maximal plane graph).
- It maximizes the minimum angle.
- It is a supergraph of the nearest neighbor graph.
- What about more involved properties? For example, is DG(S) always Hamiltonian?

Dillencourt (IPL, 1987) answered this question negatively by providing an example of a set of points whose Delaunay graph is a non-Hamiltonian triangulation.



Outline

1. Toughness of Delaunay graphs

2. Hamiltonicity of higher order proximity graphs

Does DT always satisfy a weaker condition?

Definition

For a graph G with vertex set S, and for a set $P \subseteq S$, let $c(S - P)$ be the number of components of the subgraph induced by $S \setminus P$.

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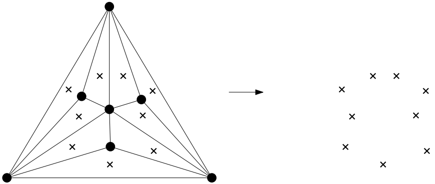

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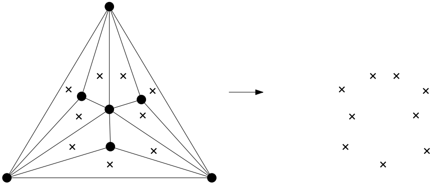

Observations

- G is Hamiltonian $\Rightarrow$ G is 1-tough

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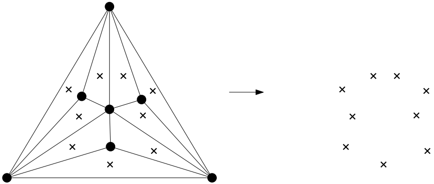

Observations

- G is Hamiltonian $\Rightarrow$ G is 1-tough
- G is 1-tough & $|S|$ is even $\Rightarrow$ G has a perfect matching

DT is always 1-tough

Theorem [Dillencourt (DCG, 1990)]

For any set S of points in the plane, $\mathrm{DT}(S)$ is 1-tough.

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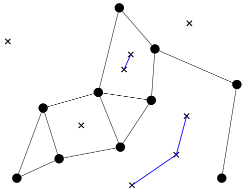

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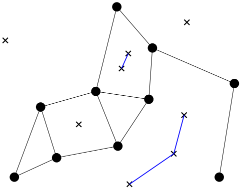
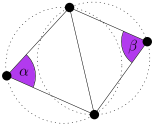
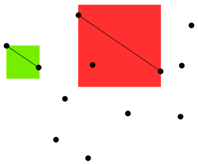


- A key property used is:



$$\alpha + \beta < \pi$$

Empty square property



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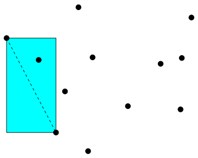

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- In this case, the convex hull edges do not necessarily belong to the graph.

The L_∞ Delaunay graph

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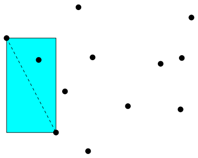

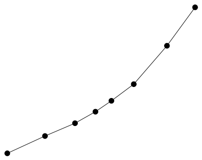

Observation

- In this case, the convex hull edges do not necessarily belong to the graph.
- We can easily find examples where the graph is not Hamiltonian.

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The L_∞ Delaunay graph of any point set S has a Hamiltonian path.

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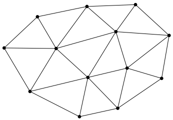

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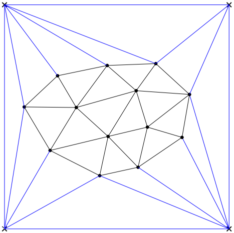

1. Consider the initial graph.
2. Add four vertices and recompute the graph.

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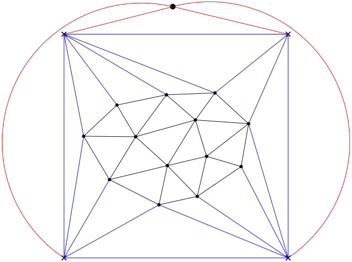

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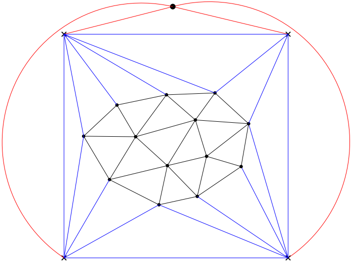

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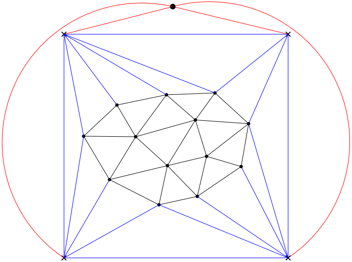

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Alternative proof for 1-toughness [Bose & S., 2010].

Questions

- The Delaunay graph with respect to the L_2 metric is 1-tough, and with respect to the L_∞ (and L_1) metric is "almost" 1-tough. Is it true that, for any $p \geq 1$, the Delaunay graph with respect to the L_p metric is 1-tough (or "almost")?

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- For which values of m is the Delaunay graph with a regular m-gon as empty region 1-tough? It is 1-tough (or “almost”) for $m = 4$ and $m = \infty$, and it is not for $m = 3$ [Bonichon, Gavoille, Hanusse & Ilcinkas (WG, 2010)].

Description

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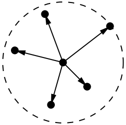
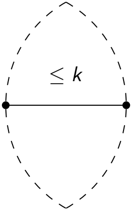
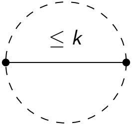
Examples



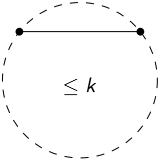
k-nearest neighbor graph (k-NNG(P))



k-relative neighborhood graph (k-RNG(P))



k-Gabriel graph (k-GG(P))



k-Delaunay graph (k-DG(P))

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Hamiltonicity of k-DG(S)

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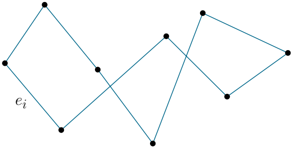
- 10-GG(S) (and thus 10-DG(S)) is always Hamiltonian [Kaiser, S. & Van Cleemput (2014)]

Proof

Let H be the set of all Hamiltonian cycles on S, and let $h \in H$.



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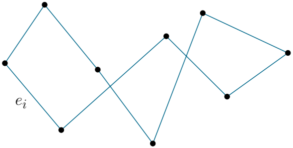

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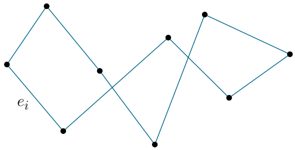

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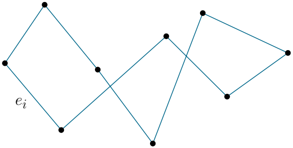

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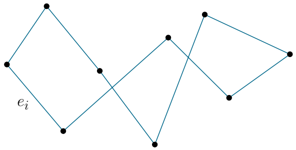

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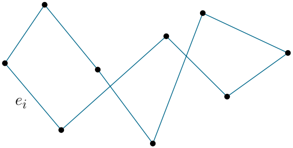

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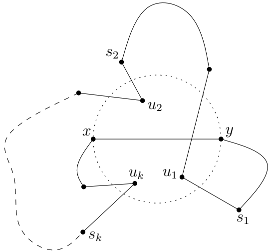

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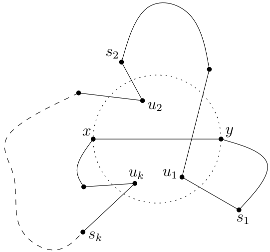

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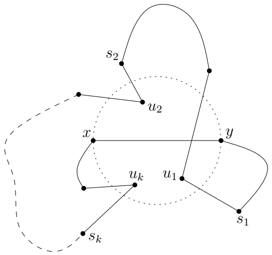

We want to prove that $k \leq 10$. First, we observe:

1. $d(s_i, x) \geq \max\{d(s_i, u_i), d(x, y)\}$ (for $1 \leq i \leq k$);
2. $d(s_i, s_j) \geq \max\{d(s_i, u_i), d(s_j, u_j), d(x, y)\}$ (for $1 \leq i < j \leq k$).

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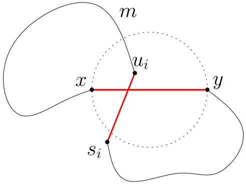

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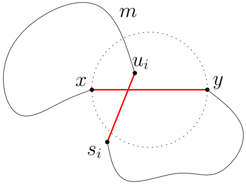

We have: $d(u_i, y) < d(x, y) \leq \max\{d(s_i, u_i), d(x, y)\}$.

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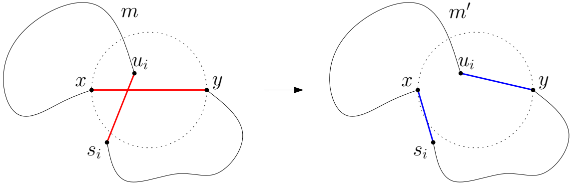

We have: $d(u_i, y) < d(x, y) \leq \max\{d(s_i, u_i), d(x, y)\}$.
If $d(s_i, x) < \max\{d(s_i, u_i), d(x, y)\}$, then

$$\max\{d(s_i, x), d(u_i, y)\} < \max\{d(s_i, u_i), d(x, y)\}.$$

Thus we would obtain that $m' < m$, a contradiction.

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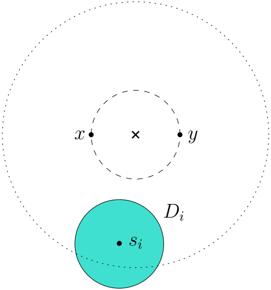
We define unit disks D_i as follows:



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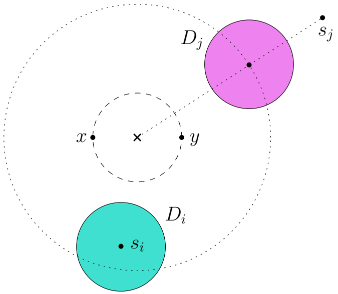

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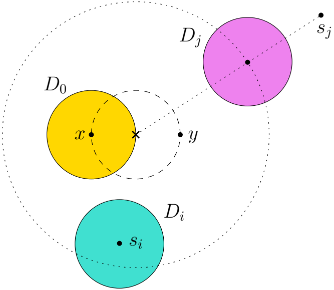

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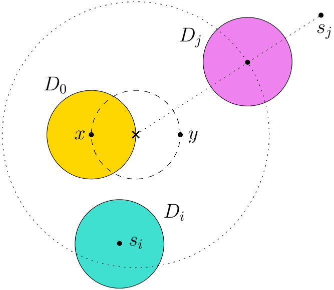

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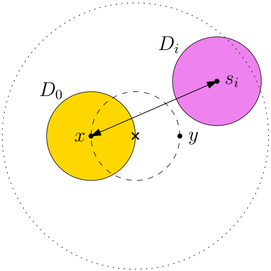

$$\begin{aligned} d(s_i, x) &\geq \max\{d(s_i, u_i), d(x, y)\} \\ &\geq d(x, y) = 2 \end{aligned}$$

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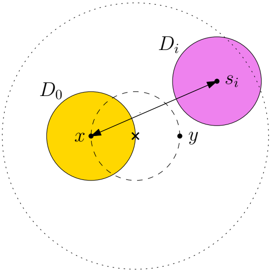

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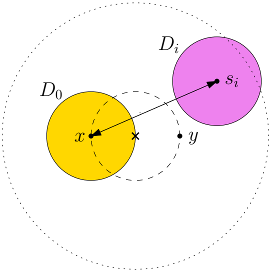

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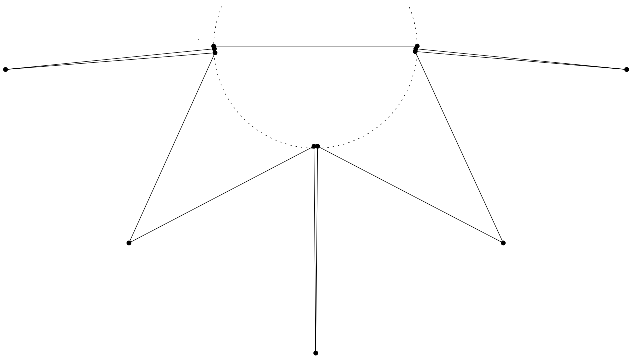
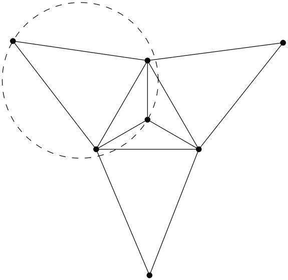
Remark

With this method, the best that one can prove is that $6\text{-GG}(S)$ is Hamiltonian:



Observation

There exist point sets S such that 1-GG(S) is not Hamiltonian:



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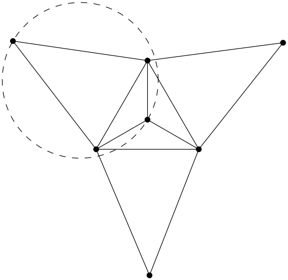

Conjecture [Abellanas, Bose, García, Hurtado, Nicolás & Ramos (IJCGA, 2009)]

1-DG(S) is always Hamiltonian.

Thank you!