# Helly type theorems for the sum of unit vectors

by I. Bárány and J. Jerónimo-Castro

Notation: $B$ unit ball of a norm $\|.\|$ in $\mathbb{R}^2$

$B$ - 0-symmetric convex body

$V \subset B$ or $V \in \partial B$ finite, $V = \{v_1, \dots, v_n\}$

**Thm 1.** $\left.\begin{array}{l} V \subset \partial B,\ |V| = n \text{ odd} \\ \underbrace{\exists u \neq 0 \text{ with } u.v_i \geq 0\ \forall i}_{\text{same as } 0 \notin \operatorname{int} \operatorname{conv} V} \end{array}\right\} \Rightarrow \left\|\sum_1^n v_i\right\| \geq 1$

Remark. $n$ odd is important

A <u>3-sum of $V$</u> is the sum of 3 distinct elements of $V$. <u>3-sum</u> for short

---

<u>Thm 2.</u> $\left.\begin{array}{l} V \subset \partial B,\ n \text{ odd}, \\ \text{every 3-sum has norm} \geqq 1 \end{array}\right\} \Rightarrow \left\| \sum_1^n v_i \right\| \geqq 1.$

---

<u>Thm 3.</u> $\left.\begin{array}{l} V \subset B,\ n \text{ odd} \\ \text{every 3-sum has norm} > 1 \end{array}\right\} \Rightarrow \left\| \sum_1^n v_i \right\| > 1.$

---

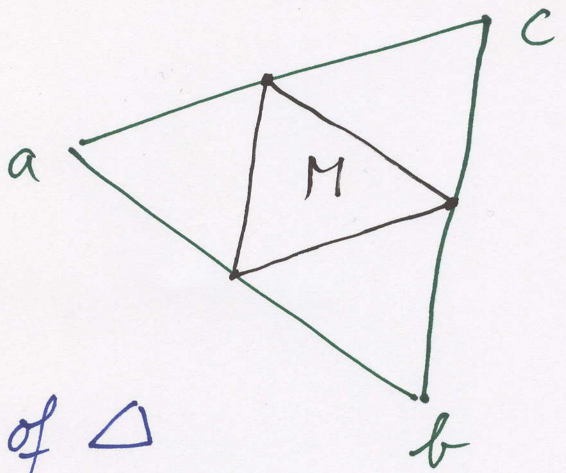
- Helly-type
- $n$ odd is important, again
- (strangely) Thm 2 fails with condition $V \subset B$ (instead of $V \subset \partial B$)
- Higher dim?

$a, b, c \in \partial B$

$h = a+b+c$

$\Delta = \operatorname{conv}\{a, b, c\}$

$M$ = medial triangle of $\Delta$




**Fact.** $h \in \Delta \iff 0 \in \Delta \iff 0 \in M$

and $0 \in \partial M \iff h \in \partial \Delta$

**Proof of Thm 2.**

Assume that $V$ contains no antipodal pair

$\Rightarrow h = v_i + v_j + v_k \notin \operatorname{int} B \quad \forall i, j, k$ distinct $\Rightarrow$

$\Rightarrow h \notin \operatorname{int} \Delta, \quad \Delta = \operatorname{conv}\{v_i, v_j, v_k\} \quad \forall i, j, k$

If $h \notin \Delta$ $\forall i, j, k$ distinct, then $0 \in \Delta$ $\forall i, j, k$ and Thm 1 applies

So Suppose $h \in \partial \Delta$ for some $\Delta = \operatorname{conv}\{v_i, v_j, v_k\}$

then $h \in \partial B$ (as $\Delta \subset B$ and $h \notin \operatorname{int} B$)

finally, if $V$ contains an antipodal pair, induction works. $\square$

---

**Lemma** (six vectors) $z_1, z_2, \dots, z_6 \in B$ and

$$\sum_1^6 z_i = 0 \Rightarrow \exists \text{ a 3-sum } \in B.$$

---

**Proof** of Thm 3 (sketch)

$n = 5$ : set $z_i = v_i$ and $z_6 = -\sum_1^5 v_i$

If $\|z_6\| \leq 1$ apply the 6-vectors lemma

---

$n \geq 7$ $V, B$ counterexample with smallest $n$

set $v_0 = -\sum_1^n v_i$ so $\|v_0\| \leq 1$

Define $D = \operatorname{conv}\{\pm v_0, \pm v_1, \dots, \pm v_n\}$

unit ball of a norm $|\ |_D$

**Fact** $V, D$ is a counterexample again

(but the worm disappears!)

$V \subset D$, every 3-sum outside $D$

$\downarrow$ change

$U \subset \operatorname{int} D$, every 3-sum of $U$ is outside $D$

**and** every 5-sum of $U$ is outside $D$

**and** $U$ is in general position

---

**and** $|5\text{-sum}|_D > |3\text{-sum}|_D$

. . .

6-vector lemma extended

(by G. Ambrus, I.B., V. Grinberg)

**Thm** $Z = \{z_1, \dots, z_n\} \subset B \subset \mathbb{R}^2$,

$\sum_1^n z_i = 0$, $k \in \{1, 2, \dots, n\}$

$\Rightarrow \exists W \subset Z$, $|W| = k$ such that

$$\sum_{z \in W} z \in B$$

**In higher dim** ... $B \subset \mathbb{R}^d$ ...

$$\Rightarrow \quad \dots \quad \sum_{z \in W} z \in \frac{d+1}{2} B$$

best possible for odd $d$

for even $d$, $\exists$ example with $\frac{d}{2} B$.