Introduction to Extended Formulations

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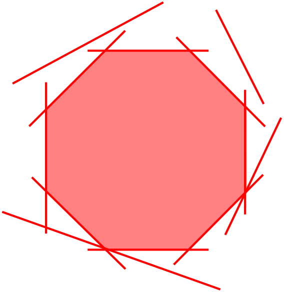
Polytopes

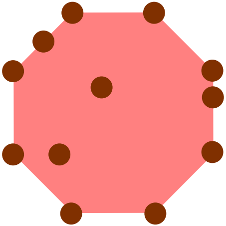
Polytope: Bounded intersection of finitely many halfspaces



$$P := \{x \in \mathbb{R}^d \mid Ax \leqslant b\}$$

Polytopes

Polytope: Bounded intersection of finitely many halfspaces
Alternatively: Convex hull of finitely many points



$$P := \{x \in \mathbb{R}^d \mid Ax \leqslant b\} = \mathrm{conv}(V)$$

Polytopes & Extended Formulations

Extended formulation: A polytope Q is an extended formulation (**EF**) of P if P is a **projection** of Q.

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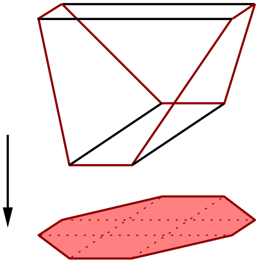

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Extension complexity denoted **ex**(P) is the **minimum** number of inequalities representing any EF of P.

Example: $xc(P_n) = \Theta(\log n)$ where P_n is a regular n-gon.

Extension Complexity

Basic Questions

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- $\mathrm{xc}(PM_n) = 2^{\Theta(n)}$ **[Rothvoß 2013]**
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$\forall n \in \mathbb{N}, L \subseteq \{0,1\}^* \ \ \exists P_1, P_2 \subseteq \mathbb{R}^{n+1}$ s.t.

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$\forall n \in \mathbb{N}, L \subseteq \{0,1\}^* \quad \exists P_1, P_2 \subseteq \mathbb{R}^{n+1}$ s.t.

$\exists P : (P_1 \subseteq P \subseteq P_2 \wedge \text{xc}(P) = \text{poly}(n)) \iff L \in$ **P/poly**

[Avis, Bremner, T., Watanabe 2014]

Extension Complexity

Slack Matrices

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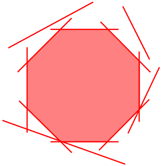

$$P = \{x \mid A_1 x \leqslant b_1, \ldots, A_m x \leqslant b_m\}$$

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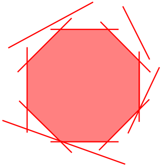

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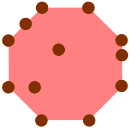

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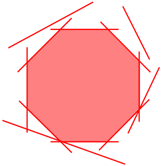

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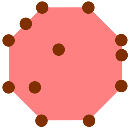

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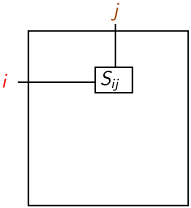
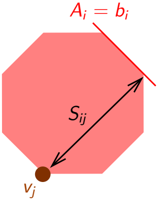
Definition

Slack matrix $S \in \mathbb{R}_+^{m \times n}$ of P: $\qquad S_{ij} := b_i - A_i v_j$





Extension Complexity

Nonnegative Factorizations

Definition

A rank-r nonnegative factorization of $S \in \mathbb{R}^{m \times n}$ is

$$S = TU \quad \text{where} \quad T \in \mathbb{R}_+^{m \times r} \quad \text{and} \quad U \in \mathbb{R}_+^{r \times n}$$

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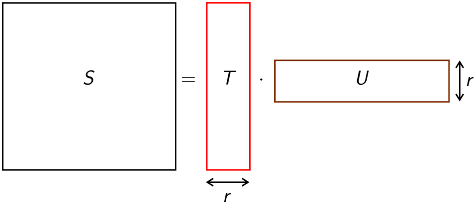

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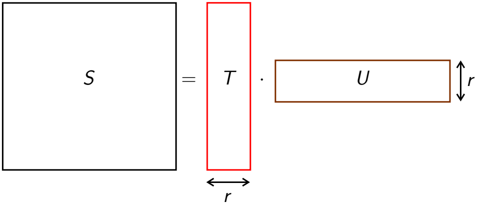


Definition

Nonnegative rank of $S \in \mathbb{R}^{m \times n}$:

$$\mathrm{rank}_+(S) := \min\{r \mid \exists \text{ rank-}r \text{ nonnegative factorization of } S\}$$

Extension Complexity

(Capuccino) Exercise!

Show that adding redundant inequalities and points to the description of a polytope **does not** change the nonnegative rank of its Slack Matrix.

(**Hint:** Just add redundant points first.)

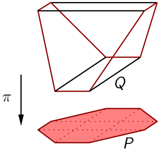
Extension Complexity

The Twofold Way

The following are equivalent [**Yannakakis**'88/91]:

1. A linear system $Ex + Fy = g,\ y \geqslant \mathbf{0}$ with $y \in \mathbb{R}^r$ s.t.

$$P = \{x \in \mathbb{R}^d \mid \exists y \in \mathbb{R}^r : Ex + Fy = g,\ y \geqslant \mathbf{0}\}$$



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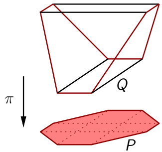

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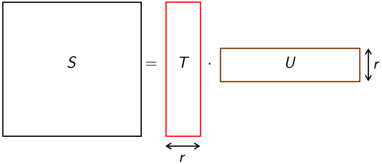

Extension Complexity: Example

Free-join of polytopes

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$$\Longrightarrow \mathrm{xc}(P_1 * P_2) \leqslant \mathrm{xc}(P_1) + \mathrm{xc}(P_2)$$

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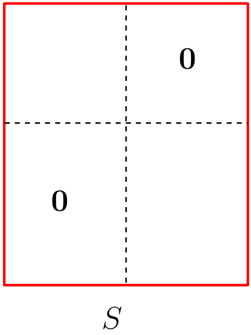

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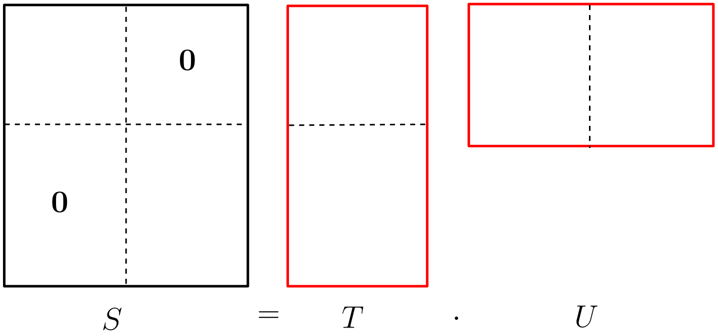

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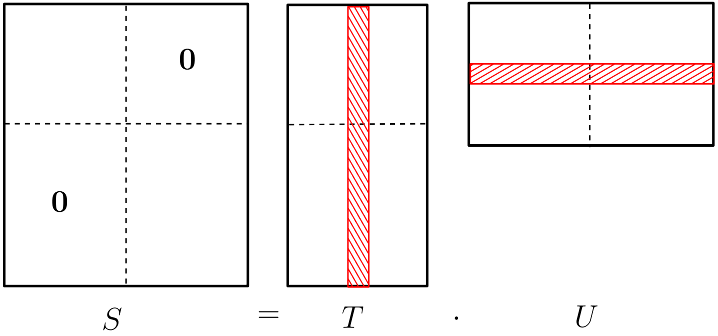

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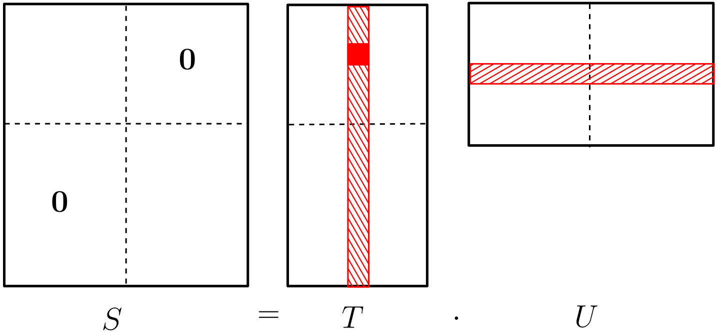

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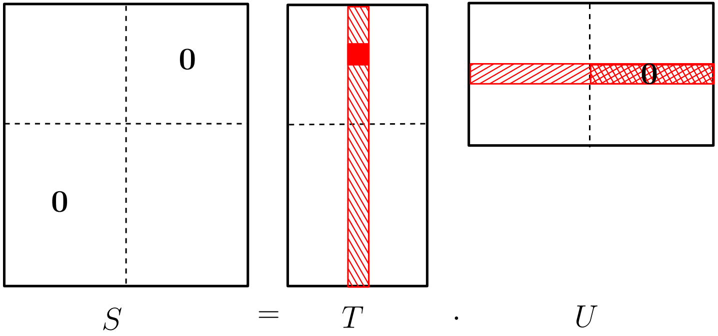

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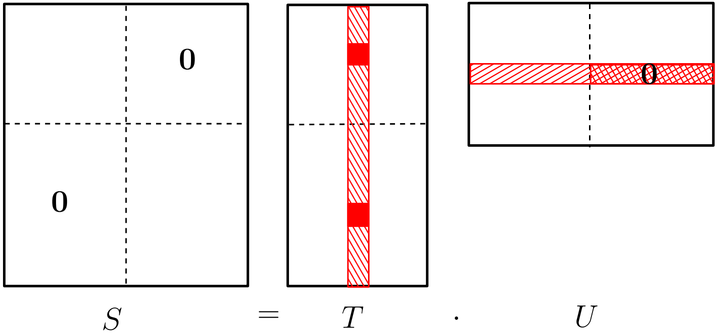

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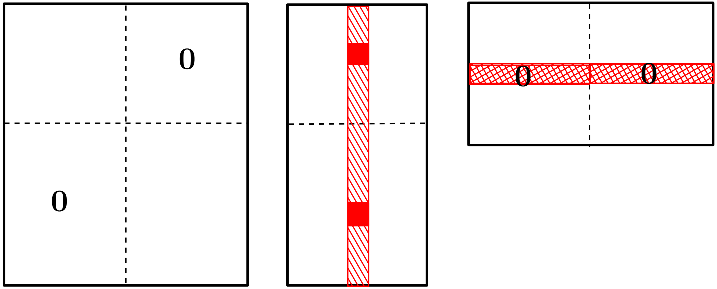

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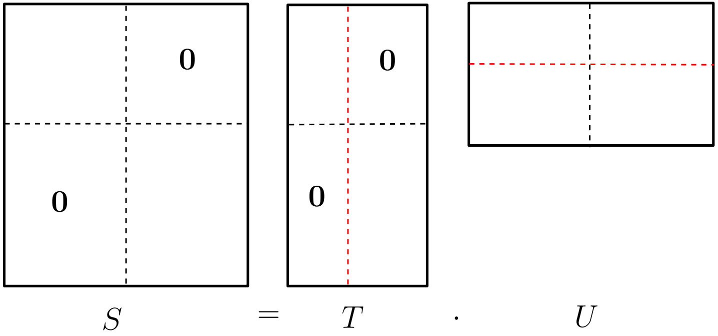

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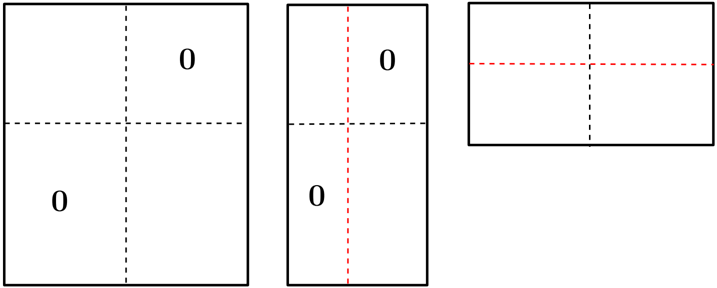

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$\Longrightarrow \text{xc}(P_1 * P_2) \geqslant \text{xc}(P_1) + \text{xc}(P_2)$

Open Problem

Product of polytopes: (Posed by François Glineur at Dagstuhl 2013)

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Is it true that $\mathrm{xc}(P_1 \times P_2) = \mathrm{xc}(P_1) + \mathrm{xc}(P_2)$?

Thank You!