An Improved Lower Bound on the Maximum Number of Noncrossing Perfect Matchings: $\Omega(3.053^n)/\mathrm{poly}(n)$

Andrei Asinowski, Günter Rote
Freie Universität Berlin

General position: no three points on a line

Background

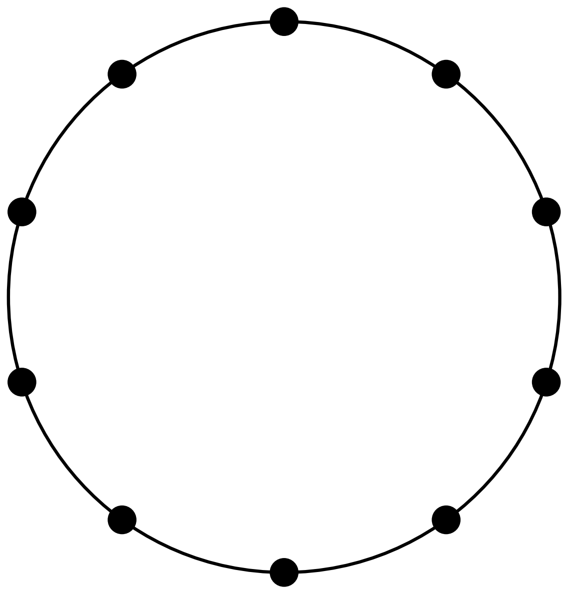
Given a set of n points in the plane in general position, how many

- triangulations
- non-crossing spanning trees
- non-crossing Hamiltonian cycles
- non-crossing matchings
- non-crossing perfect matchings $\leftarrow$ THIS TALK
- . . .
- *[your favorite straight-line geometric graph structure]*

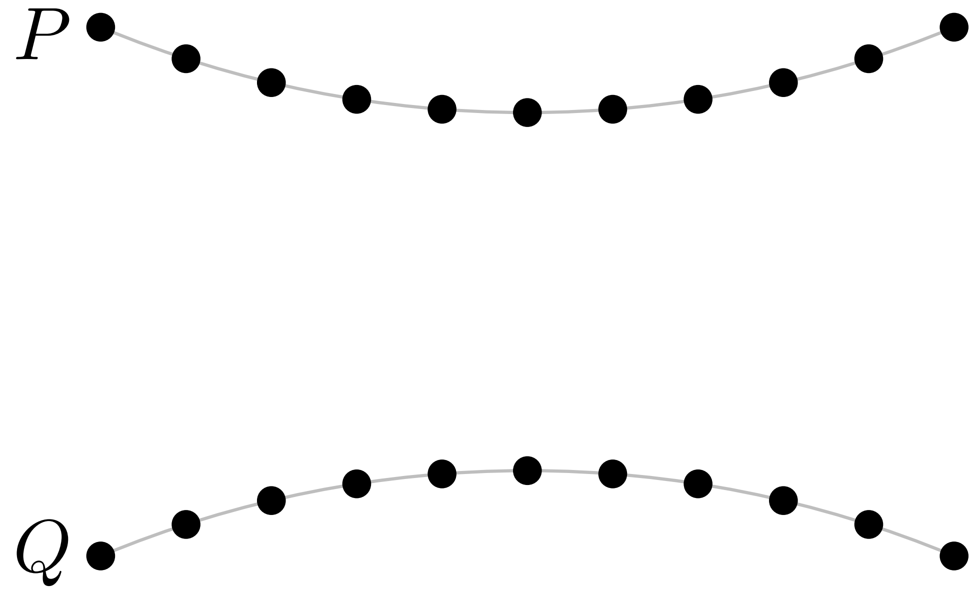
can it have?

Previous Results on Perfect Matchings

Freie Universität Berlin



convex position



double-chain

smallest possible number of perfect matchings: $\Theta^*(2^n)$

previous record: $\Theta^*(3^n)$

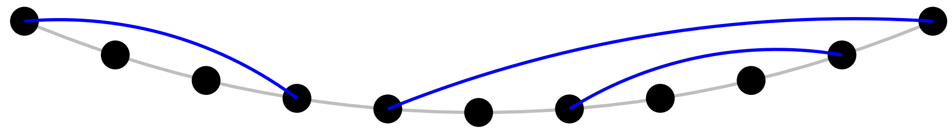
[García, Noy, Tejel 2000]

Upper bound: $O^*(10.06^n)$ [Sharir, Welzl 2006]

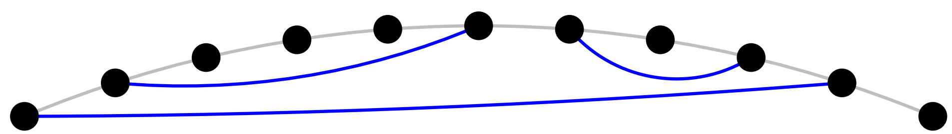
O^*, Θ^* = up to a polynomial factor

Simplified Analysis of the Double-Chain

P

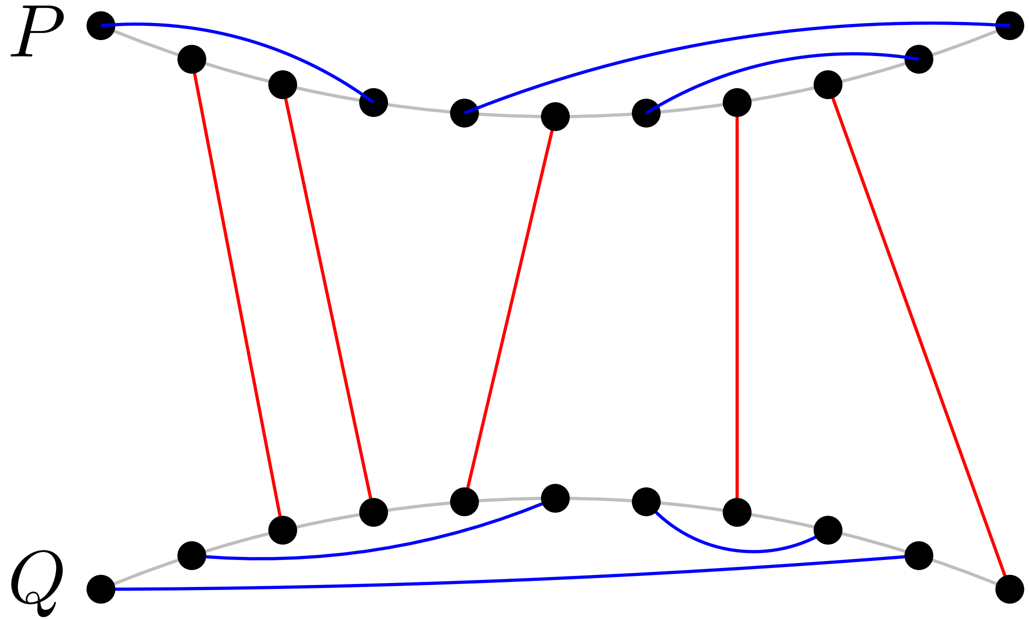


Q



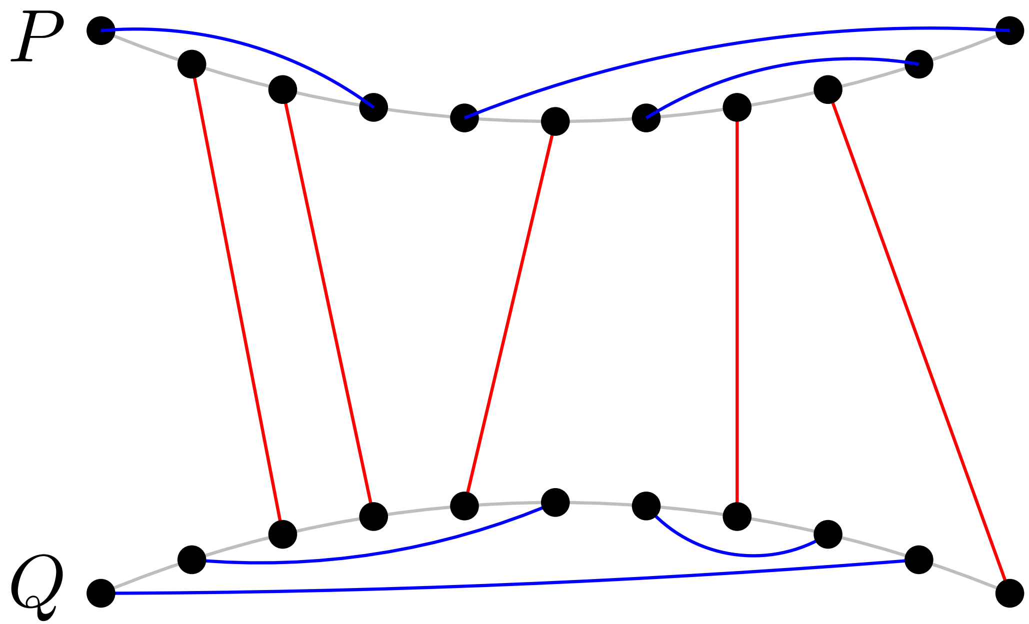
Any matching of P with k free points and any matching of Q with k free points can be extended to a unique perfect matching.

Simplified Analysis of the Double-Chain



Any matching of P with k free points and any matching of Q with k free points can be extended to a unique perfect matching.

Simplified Analysis of the Double-Chain



Any matching of P with k free points and any matching of Q with k free points can be extended to a unique perfect matching.

$M = \#$(matchings in P); M_k matchings have k free nodes.

$$T = \#(\text{perfect matchings in } P \cup Q) = \sum_k (M_k)^2$$

Simplified Analysis

Freie Universität Berlin

$M = \#(\text{matchings in } P)$;
$M_k = M p_k$ matchings have k free nodes. $\sum_k p_k = 1$.

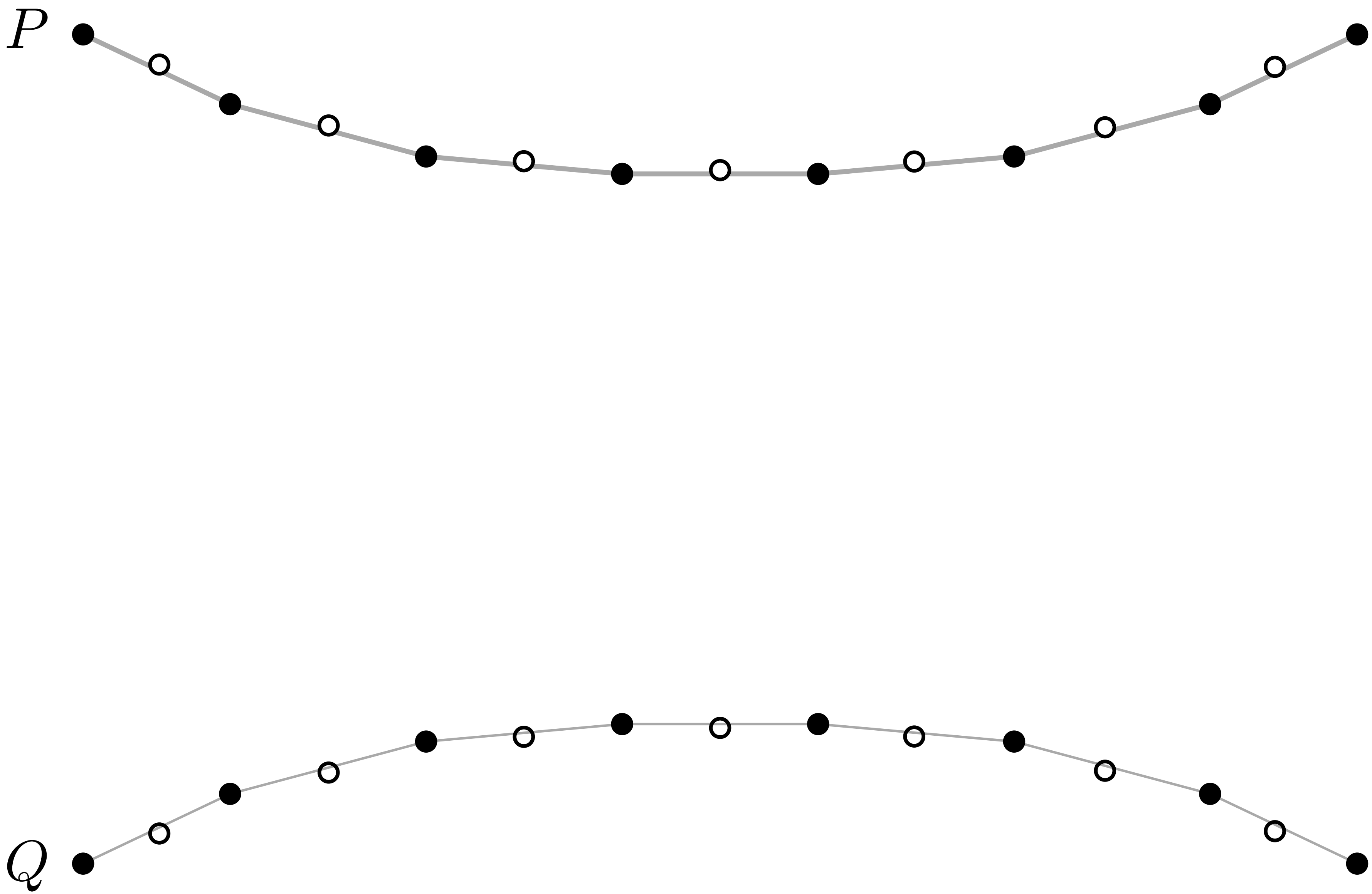
$$T = \sum_k (M_k)^2 = M^2 \sum_k p_k^2$$

$$\frac{1}{n} \le \sum_{k=1}^{n} p_k^2 \le 1$$

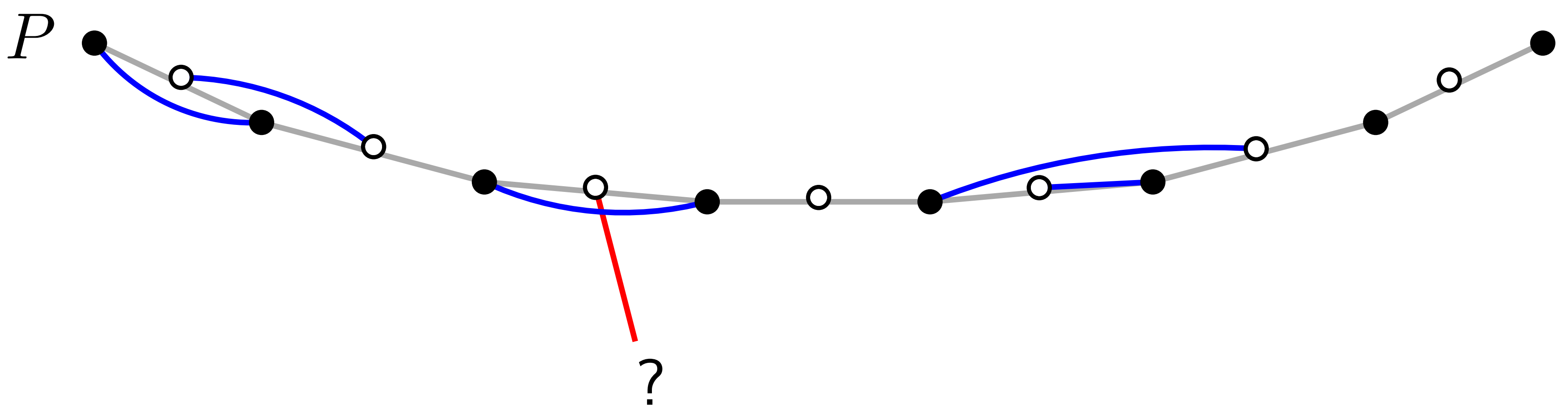
$M =$ Motzkin numbers $= \Theta(3^{n/2} n^{-3/2}) \implies T = \Theta^*(3^n)$

It suffices to count (not necessarily perfect) matchings of P!

The Double-Zigzag Chain

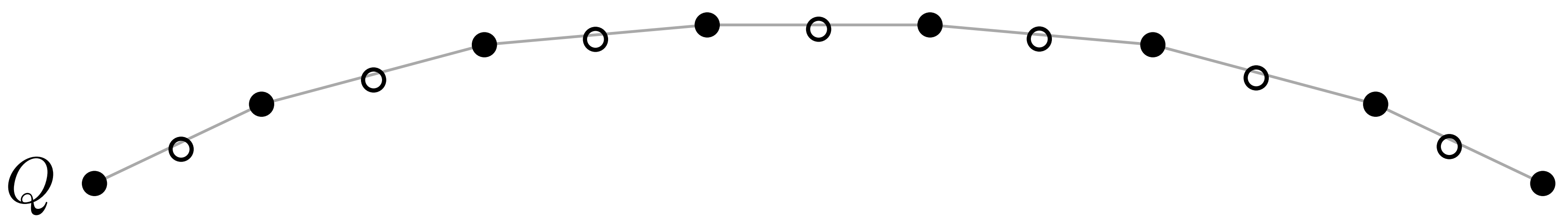


The Double-Zigzag Chain

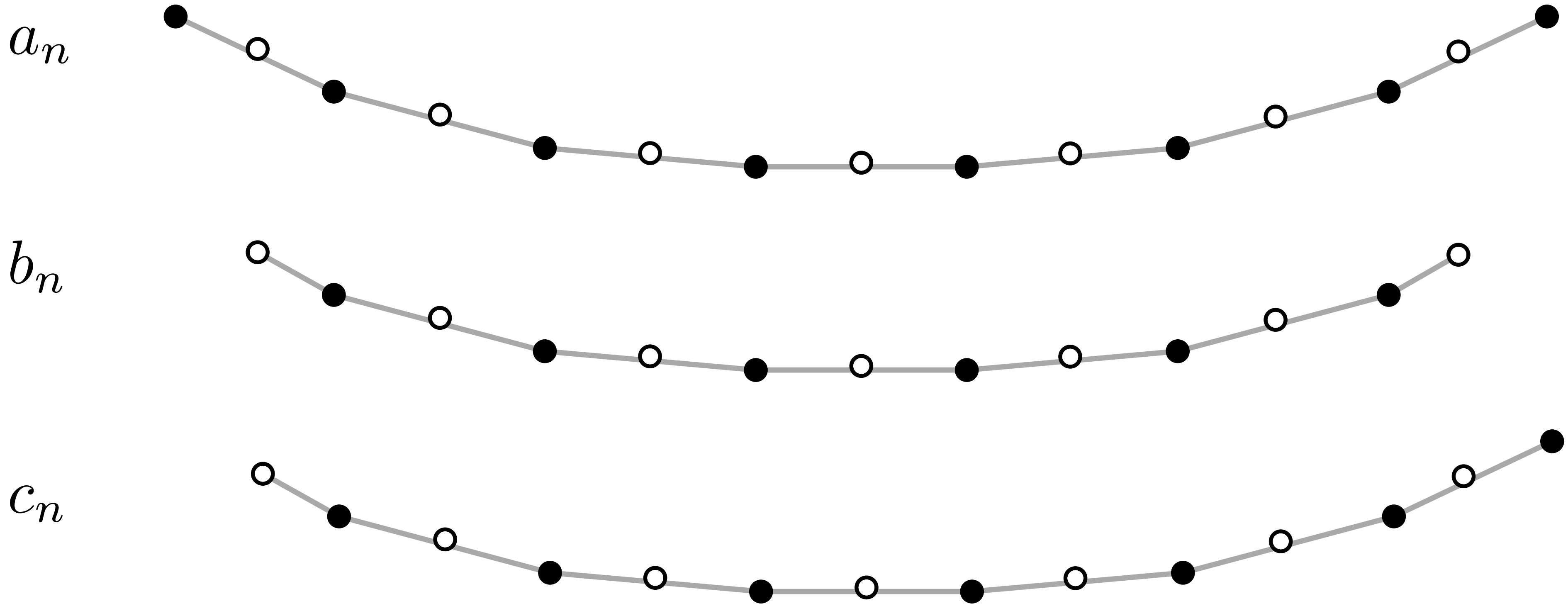


Not every matching of P works.

The free vertices must be visible from below: *down-free*



Counting Down-Free Matchings

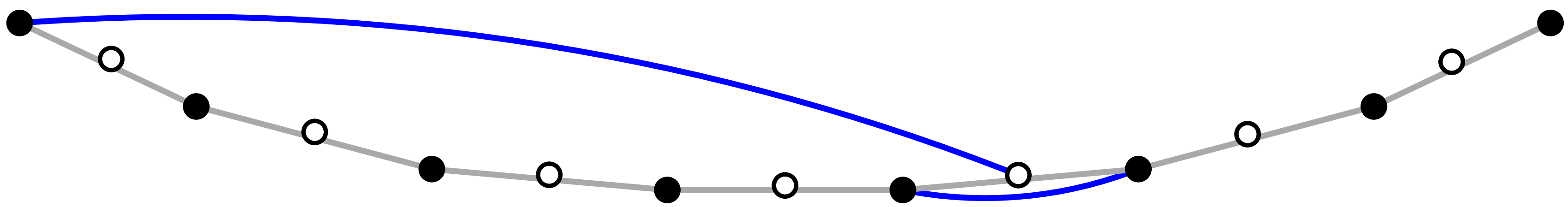


$$a_n, b_n, c_n = \#\text{ of down-free matchings}$$

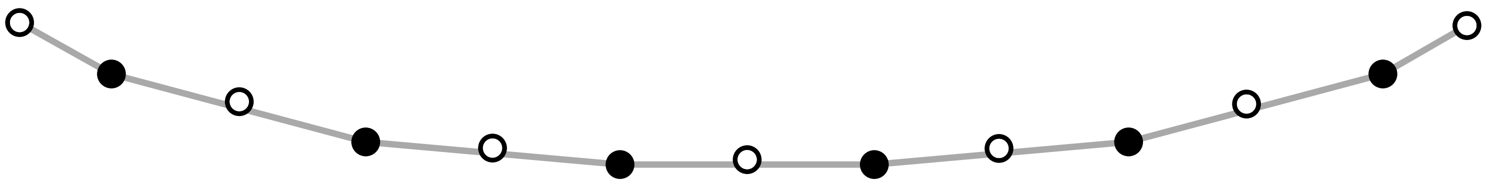
Counting Down-Free Matchings

Freie Universität Berlin

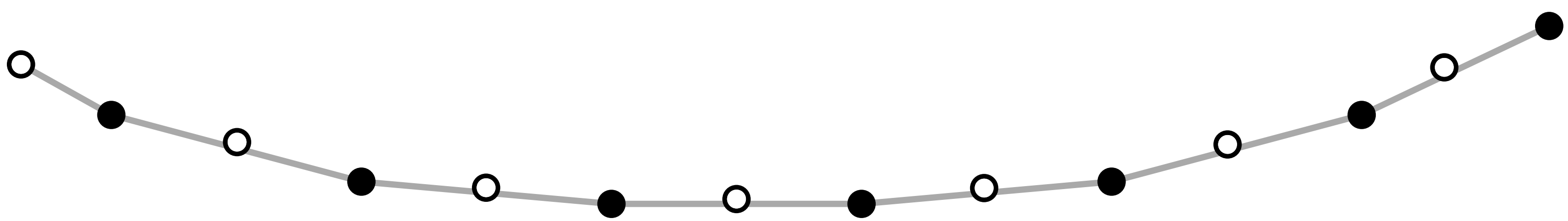
a_n



b_n



c_n



$a_n, b_n, c_n = \#$ of down-free matchings

$$a_n = c_n + \sum_i b_i c_{n-1-i} + \cdots$$

$$b_n = \cdots$$

Analysis with Formal Power Series

$A = A(x) = \sum_n a_n x^n$, etc.

$$A = C((1-x) + x(1+x)A + x(1+x)^2 B)$$

$$B = C(1 + xA + x(1+x)B)$$

$$C = 1 + xA + x^2 A^2 + x(1+x)C^2$$

Analysis with Formal Power Series

$A = A(x) = \sum_n a_n x^n$, etc.

$$A = C((1-x) + x(1+x)A + x(1+x)^2 B)$$
$$B = C(1 + xA + x(1+x)B)$$
$$C = 1 + xA + x^2A^2 + x(1+x)C^2$$

$$C = \frac{2(1+x+x^3) - \sqrt{2(1+x+x^3)\left(1 - 2x - 8x^2 - 3x^3 + (1+x)\sqrt{(1-x-3x^2)(1-9x-3x^2)}\right)}}{4x(1+x)(1+x+x^3)}$$

$$A = \frac{C(1 - x + 2x^2C + 2x^3C)}{1 - 2xC - 2x^2C}$$

$$B = \frac{C(1 - 2x^2C)}{1 - 2xC - 2x^2C}$$

The Proof

$$C = \frac{2(1+x+x^3) - \sqrt{2(1+x+x^3)\left(1-2x-8x^2-3x^3+(1+x)\sqrt{(1-x-3x^2)(1-9x-3x^2)}\right)}}{4x(1+x)(1+x+x^3)}$$

smallest singularity: $1 - 9x - 3x^2 = 0$

$$x_0 = \frac{\sqrt{93}}{6} - \frac{3}{2}$$

$$1/\sqrt{x_0} = \sqrt{6/(\sqrt{93}-9)} \approx 3.0532$$

$$\boxed{\#(\text{perfect matchings in } P \cup Q) = \Theta^*(3.0532^n),}$$

where $n = |P \cup Q|$.

Longer Arcs

$|P| = nr$

Longer Arcs

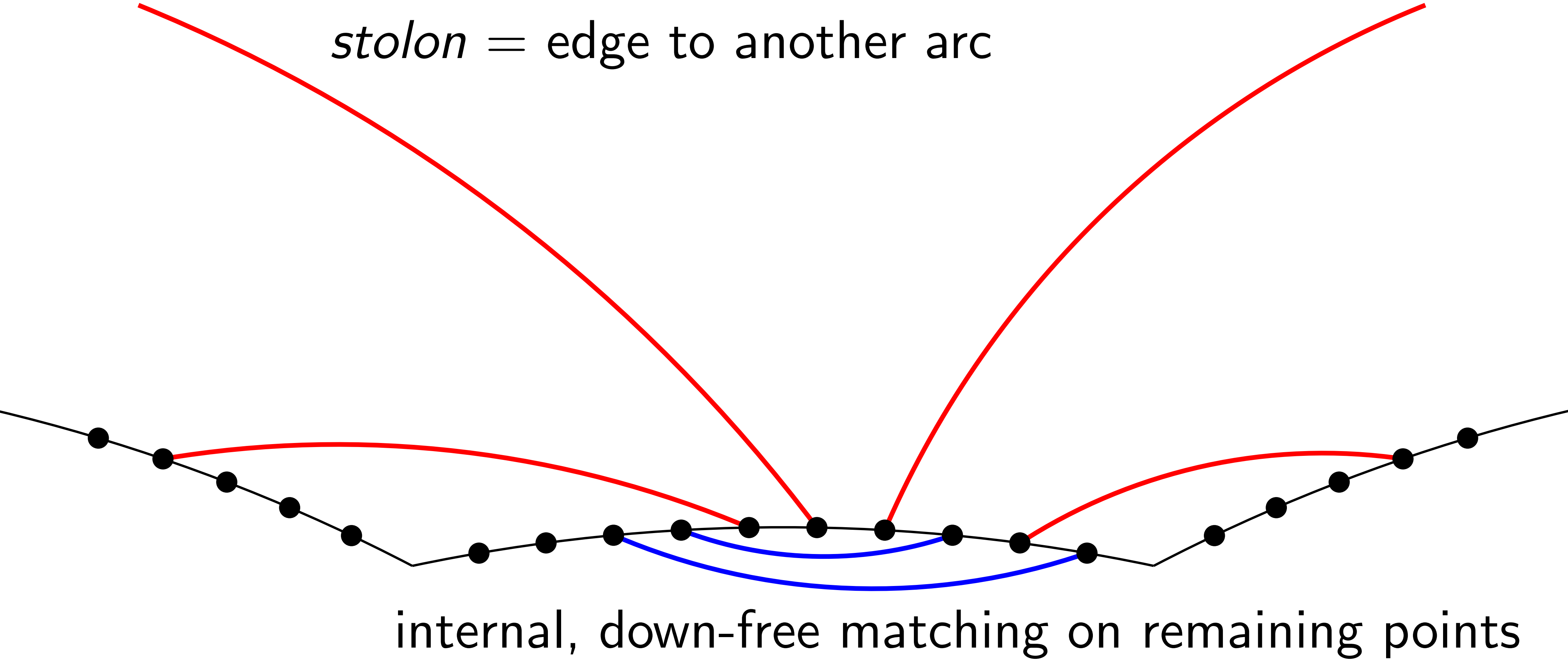
$|P| = nr$

Longer Arcs

$|P| = nr$

Omit the corner points $\rightarrow$ easier recursion

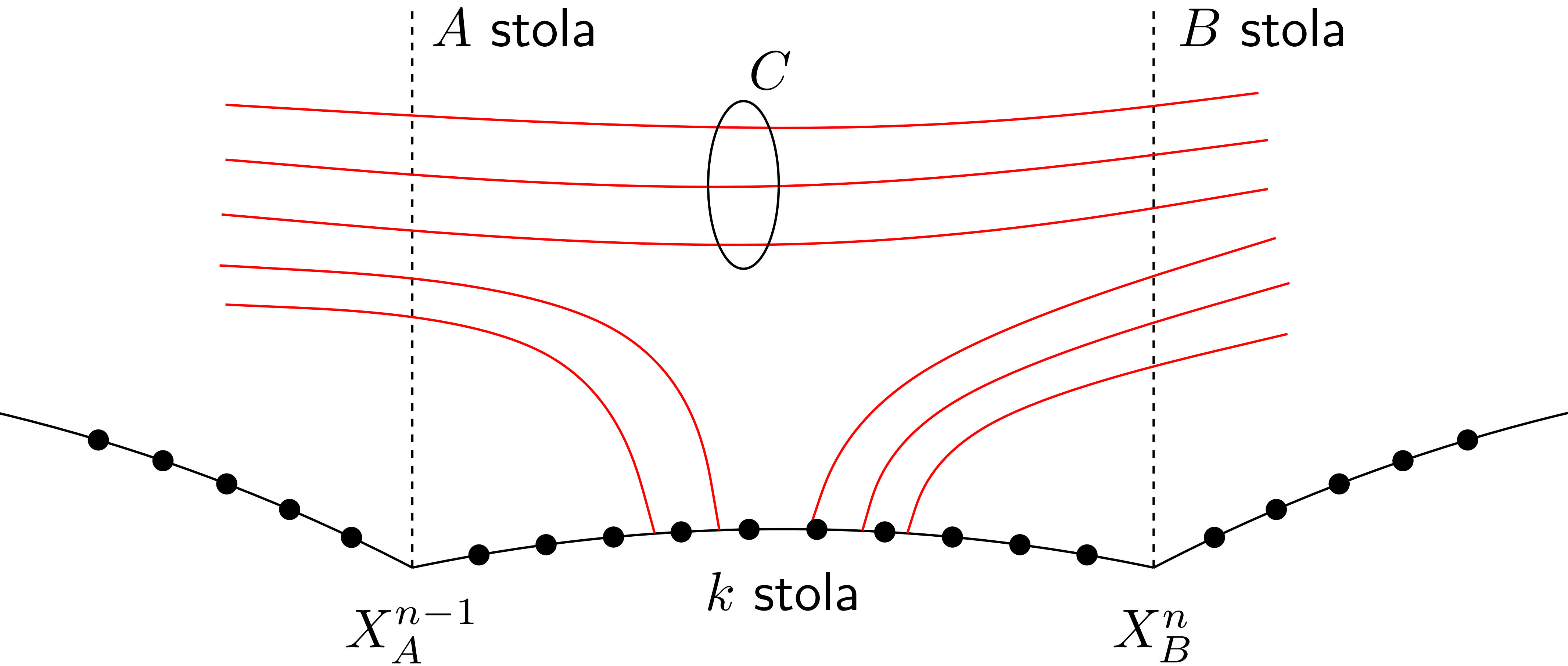
Dynamic Programming Recursion



k stola: $\binom{r}{k}\binom{r-k}{\lfloor (r-k)/2 \rfloor}$ possibilities

down-free matching on $r-k$ points

Dynamic Programming Recursion



X_B^n = # possibilities after n arcs with B crossing stola.

... try all choices of k and C: $0 \le k \le r$, $0 \le C \le B$

$$A = C + (k - (B - C))$$

Example: $r = 5$

matrix for transforming $(X_0^{n-1}, X_1^{n-1}, X_2^{n-1}, \ldots)$ into $(X_0^n, X_1^n, X_2^n, \ldots)$

$$\begin{pmatrix}
10 & 30 & 30 & 20 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
30 & 40 & 50 & 35 & 21 & 5 & 1 & 0 & 0 & 0 & 0 & \cdots \\
30 & 50 & 45 & 51 & 35 & 21 & 5 & 1 & 0 & 0 & 0 & \cdots \\
20 & 35 & 51 & 45 & 51 & 35 & 21 & 5 & 1 & 0 & 0 & \cdots \\
5 & 21 & 35 & 51 & 45 & 51 & 35 & 21 & 5 & 1 & 0 & \cdots \\
1 & 5 & 21 & 35 & 51 & 45 & 51 & 35 & 21 & 5 & 1 & \cdots \\
0 & 1 & 5 & 21 & 35 & 51 & 45 & 51 & 35 & 21 & 5 & \cdots \\
0 & 0 & 1 & 5 & 21 & 35 & 51 & 45 & 51 & 35 & 21 & \cdots \\
0 & 0 & 0 & 1 & 5 & 21 & 35 & 51 & 45 & 51 & 35 & \cdots \\
0 & 0 & 0 & 0 & 1 & 5 & 21 & 35 & 51 & 45 & 51 & \cdots \\
0 & 0 & 0 & 0 & 0 & 1 & 5 & 21 & 35 & 51 & 45 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

row sum $= 271 \implies$ vectors grow like $271^n/\text{poly}(n)$

The Record: 3.084^n

r	exponent
1	3.0
2	3.0
3	3.03658897188
4	3.05407580998
5	3.06615325393 $= \sqrt[5]{271}$
6	3.07353334449
7	3.07825838546
8	3.08116216736
9	3.08286438954
10	3.08373678000
11	3.08403284879 $= \sqrt[11]{240054}$
12	3.08392263613
13	3.08352460563
14	3.08292219318

With Corners (Numerical Results)

$|P| = nr + 1$

more complicated recursion:
8 cases

$r = 8$: 3.0924

$\|P\|$	$\sqrt[8]{\text{quotient}}$
⋮	⋮
7921	3.0924200
7929	3.0924206
7937	3.0924211
7945	3.0924217
7953	3.0924223
7961	3.0924229