Many triangulated odd-spheres

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Q: $S_d(n) := \#\left\{\begin{array}{l}\text{combinatorially distinct} \\ n\text{-vertex triangulations} \\ \text{of the } d\text{-sphere, } S^d\end{array}\right\}$

$S_d(n) = ?$ (d fixed, n large)

UPPER BOUND: (Kalai '88):

$|\Delta| \cong S^d$

$f_i(\Delta) := \{F \in \Delta : |F| = i+1\}$

UBT for spheres

(or just Dehn-Sommerville relations)

$$\Downarrow$$

$$f_d(\Delta) = O\left(n^{\lceil \frac{d}{2} \rceil}\right)$$

$$\Downarrow$$

$$S_d(n) \le \binom{\binom{n}{d+1}}{O\left(n^{\lceil \frac{d}{2} \rceil}\right)} = 2^{O\left(n^{\lceil \frac{d}{2} \rceil} \lg n\right)}$$

LOWER BOUND:

Kalai '88, "Many triangulates spheres":

$$S_d(n) \ge \#\text{ squeezed spheres} \ge 2^{\Omega\left(n^{\lfloor \frac{d}{2} \rfloor}\right)}$$

$$\left(\Rightarrow 2^{\Omega(n)} \le S_3(n) \le 2^{O(n^2 \lg n)} \right)$$

Pfeifle–Ziegler '04, "Many triangulated 3-spheres":

$$S_3(n) \geq 2^{\Omega(n^{5/4})}$$

Idea: - construct Γ = polyhedral 3-sphere,
with n vertices,
$\Theta(n^{5/4})$ octahedra,

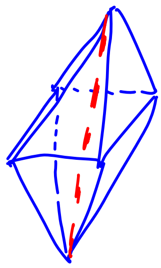
$\Rightarrow$

- can triangulate the octahedra,
by inserting a diagonal, independently.

$\Rightarrow$

- compute:

$$S_3(n) \geq \frac{1}{n!} \cdot 3^{cn^{5/4}} = 2^{\Omega(n^{5/4})}.$$



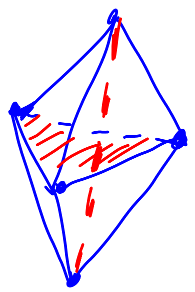
N.–Santos–Wilson '13–'14, "Many triangulated odd spheres"

Let $d = 2k-1$ odd. $\Rightarrow$ $S_{2k-1}(n) \geq 2^{\Omega(n^k)}$

(so: $S_3(n) \geq 2^{\Omega(n^2)}$.)

Idea: construct n-vertex polyhedral 3-sphere with $\Theta(n^2)$ bipyramids,

$\Rightarrow$ can triangulate each bipyramid independently in two ways.



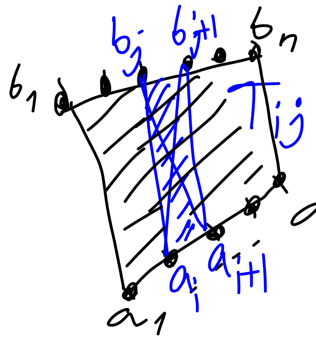
$$\Rightarrow \quad S_3(n) \geq \frac{1}{n!}\, 2^{\Omega(n^2)} = 2^{\Omega(n^2)}.$$

CONSTRUCTION:

T = join of two paths $\cong$ 3-ball:

$a_i = (i, 0, -1)$

$b_j = (0, j, +1)$

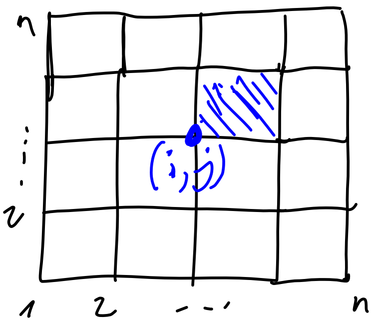


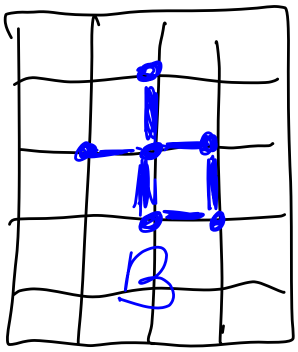
$T_{ij} = \mathrm{conv}(a_i, a_{i+1}, b_j, b_{j+1})$

Planar representation:

cut with $(z=0)$ plane:

$(i,j) \longleftrightarrow T_{ij}$

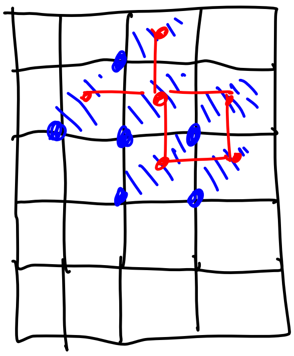




grid graph $\supseteq B \longleftrightarrow T_B =$ union of tetrahedra.

① B grid-connected $\iff \operatorname{int}(T_B)$ is connected

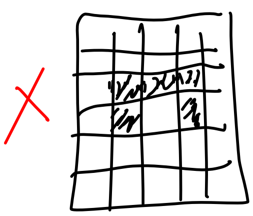
(pf: just look on the dual graph.)

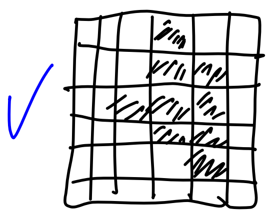


② B grid unimodal $:=$ (i) B grid connected
(ii) all fibers are connected:

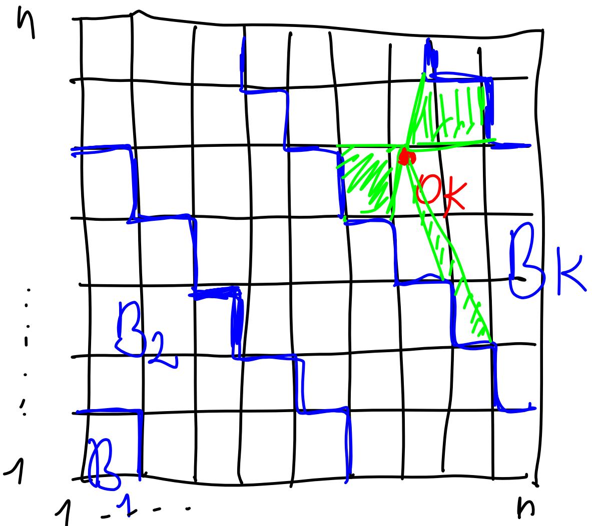
$\forall i$: $B \cap \{i\} \times [n] = \{i\} \times [j_1, j_2]$

$\forall j$ $B \cap [n] \times \{j\} = [i_1, i_2] \times \{j\}$





B grid unimodal $\iff T_B$ a (shellable) 3-ball.

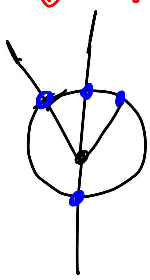


- remove int(B_K) from each 3-ball B_K.
- cone with O_K over the intersection of each tetrahedron with ∂B_K. [show: get a polyhedral 3-ball.]
- in GREEN we got bipyramidal cells, $\Theta(n^2)$ of them.

VARIATION: How many n-vertex geodesic 3-spheres?

equivalently: complete fans in $\mathbb{R}^4$.

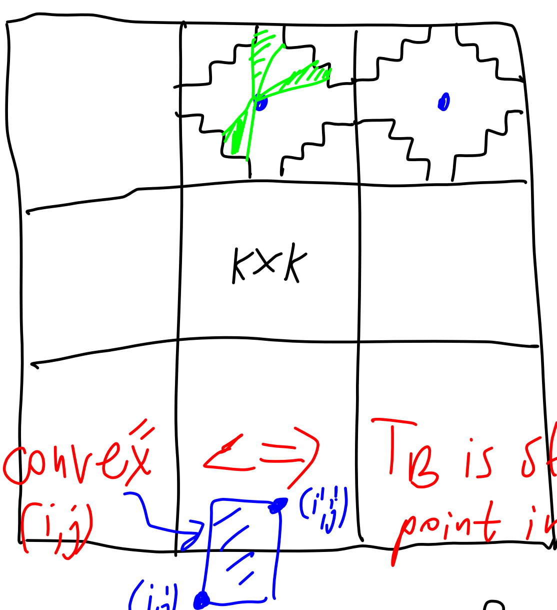
$\mathbb{R}^2$-example:



Idea: regular polyhedral 3-ball.

$\ell\times\ell$ coarse grid

$K\times K$ fine grid



is liftable ($\equiv$ regular).

! B "grid-star convex" from (i,j) $\Longleftrightarrow$ T_B is star convex from a/any point in $\operatorname{int}(T_{ij})$

compute: $K=\ell \Rightarrow$ n vertices & $\Theta(n^{3/2})$ bipyramids.

Corollary:

1. $\exists$ n-vertex 4-polytope with $\Omega(n^{3/2})$ bipyramidal faces.
2. $\exists$ at least $2^{\Omega(n^{3/2})}$ n-vertex geodesic spheres.

Q: (upper bound on NON-simplicial faces):

P 4-polytope on n vertices $\overset{?}{\Longrightarrow}$ $f_3(P) = o(n^2)$.

Q: what about the $\lg n$ factor?: $n^2 \le \lg s_3(n) \le n^2 \lg n$

(*) The above ideas generalize to high odd-dim.

Thank you!