Embeddability of collapsible complexes

Bruno Benedetti

joint work with Karim A. Adiprasito (KAA), Frank H. Lutz (FHL)

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- A *leaf* in a graph is a vertex belonging to only one edge. More generally, a *free face* in a simplicial complex is a face belonging to only one other face.

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References: arXiv:1404.4239, arXiv:1403.5217

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- **Proof that there's 2:** By induction on no. of edges. Removing a leaf from a tree, yields a tree with one edge less! (Reattaching the leaf may kill another leaf or not, so the total number of leaves is either unchanged or $+1$.)

Four properties that, when applied to graphs, mean 'tree'

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For 1-complexes they're all $\Leftrightarrow$: In acyclic connected 1-complexes (aka trees) you can recursively delete one leaf. So (1) $\Rightarrow$ (4).
For 2-complexes, however, all implications are strict.

Attempted generalization.

Every $\left\{ \begin{array}{c} \text{acyclic?} \\ \text{contractible?} \\ \text{collapsible?} \\ \text{nonevasive?} \end{array} \right.$ complex has at least 2 free faces.

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- The **Dunce Hat** (Zeeman, 1960) is an acyclic, contractible 2-complex that has no free edge.

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- The **Dunce Hat** (Zeeman, 1960) is an acyclic, contractible 2-complex that has no free edge. (In particular it is not collapsible.)
- This can be extended to all dimensions. So for contractible and for acyclic d-complexes, the trivial bound 'there are at least 0 free faces' is best possible.

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Proposition ([KAA–BB–FHL] for $d \geq 3$, [Björner] for $d = 2$)

For every $d \geq 2$, one can construct a collapsible simplicial d-complex with $2^d + d + 1$ vertices that has only 1 free face.

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Theorem [KAA–BB–FHL]

Every nonevasive complex has at least 2 free faces, and in each dimension, the bound is sharp.

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Proof: By induction on the dimension d,

- (basis) the statement is true for $d = 1$ (trees have 2 leaves);
- (step) if F and G are two free faces in $\mathrm{link}(v, C)$, which has dimension $d - 1$, then $v * F$ and $v * G$ are free faces in C.

The nontrivial part is showing sharpness (i.e. constructing a nonevasive d-complex with exactly 2 free faces for each d).

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- choosing $f(d) = 2d - 1$ or lower, we get a false statement,
- choosing $f(d) = 2d + 1$ or higher, we get a statement trivially true for **all** d-complexes,
- so the only reasonable guess is $f(d) = 2d$.

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  So subdviding it barycentrically, we even get a nonevasive d-complex that does not embed in $\mathbb{R}^{2d-1}$.

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Well known fact

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With the same proof:

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Any d-complex (contractible or not!) embeds in $\mathbb{R}^{2d+1}$, just by placing vertices in generic points.

- If $a_0, \ldots, a_d, b_0, \ldots, b_d$ are $2d + 2$ generic points of $\mathbb{R}^{2d+1}$, the two d-dimensional simplices spanned by $[a_0, \ldots, a_d]$ and $[b_0, \ldots, b_d]$ are skew to one another and disjoint.

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- This is open for contractible/acyclic 2-complexes (ongoing work with KAA). Probably false, since a much weaker conjecture, namely, that every contractible 2-complex PL embeds in $\mathbb{R}^4$, is a deep open problem, connected to the 4-dimensional smooth Poincaré conjecture.
- However...

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$$v_1 - w_1 \gg |v_i - w_i| > 0 \quad \text{for all } i \neq 1.$$

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$$v_1 - w_1 \gg |v_i - w_i| > 0 \quad \text{for all } i \neq 1.$$

With this 'cleverly generic' choice of coordinates, one can verify that all faces are embedded. □

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From these two relatively easy results, one can get more interesting things. For example:

Consequence 1. Optimal Morse vectorS [KAA–BB–FHL]

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While smooth manifolds have a (unique!) minimal Morse vector, simplicial complexes may have more than one minimal discrete Morse vector.

Proof idea: Let C_{d+1} denotes the collapsible $(d+1)$-complex with only one free face, σ_d (of dim. d).

Consequence 1. Optimal Morse vectorS [KAA–BB–FHL]

While smooth manifolds have a (unique!) minimal Morse vector, simplicial complexes may have more than one minimal discrete Morse vector.

Proof idea: Let C_{d+1} denotes the collapsible $(d+1)$-complex with only one free face, σ_d (of dim. d). We glue C_{d+1} to C_d by identifying $\sigma_d \in C_{d+1}$ with the d-face of C_d containing σ_{d-1}.

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$$(1, 0, \ldots, 0, 1, 1) \quad \text{and} \quad (1, 0, \ldots, 0, 1, 1).$$

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For $C_2 \# C_3$, explicitly, we get a 3-complex with f-vector $(106, 596, 1064, 573)$ that has both $(1, 0, 1, 1)$ and $(1, 1, 1, 0)$ as **minimal** discrete Morse vectors.

Consequence 2. PL embeddings of CAT(0) 2-complexes [KAA–BB]

The barycentric subdivision of every d-dimensional CAT(0) cube complex, embeds in $\mathbb{R}^{2d}$.

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Consequence 3. A 'weaker Tverberg', but in a complex [KAA–BB]

Let X be a d-dimensional simplicial complex, with a metric of curvature ≤ 0. Any set of $n \geq (r-1)(2d+1)+1$ points in X can be partitioned into r subsets whose convex hulls intersect.