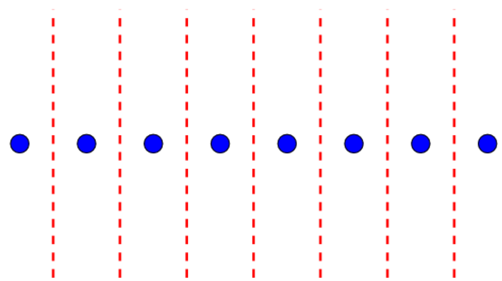
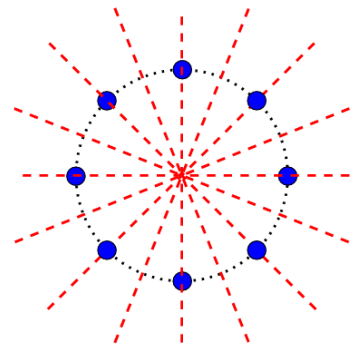
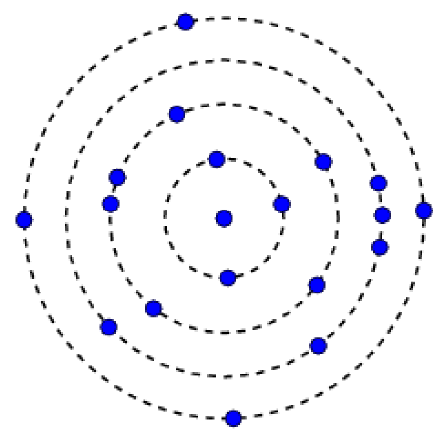
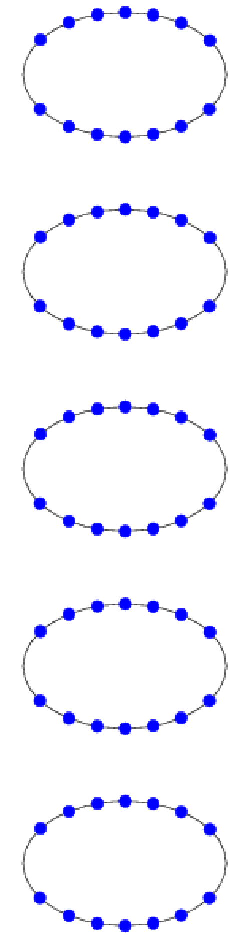
Perpendicular Bisectors and Few Distinct Distances









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Joint Work with…

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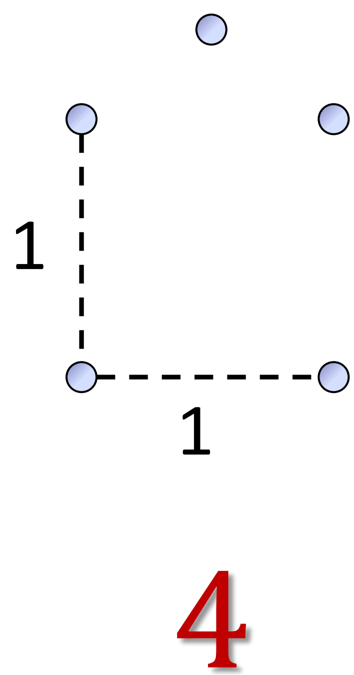
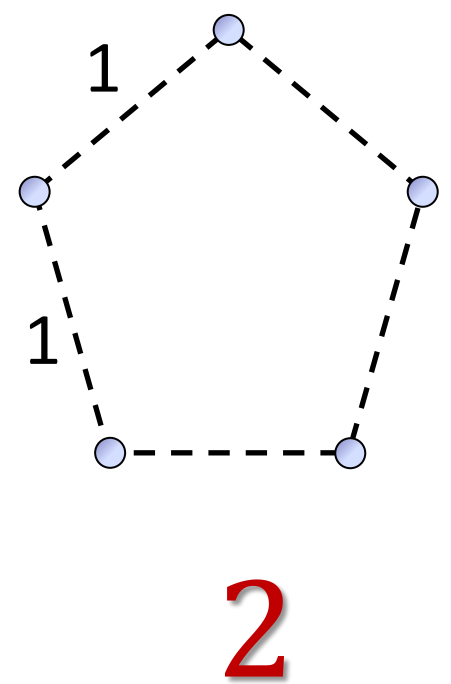
Distinct Distances

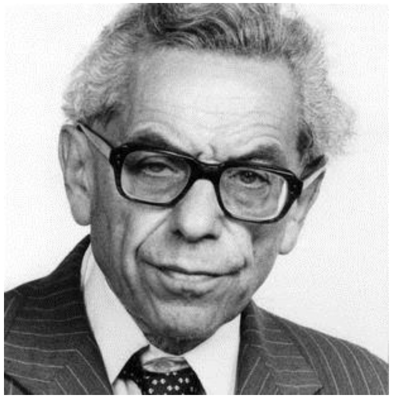
- How many *DD* (*D*istinct *D*istances) are determined by pairs of points?

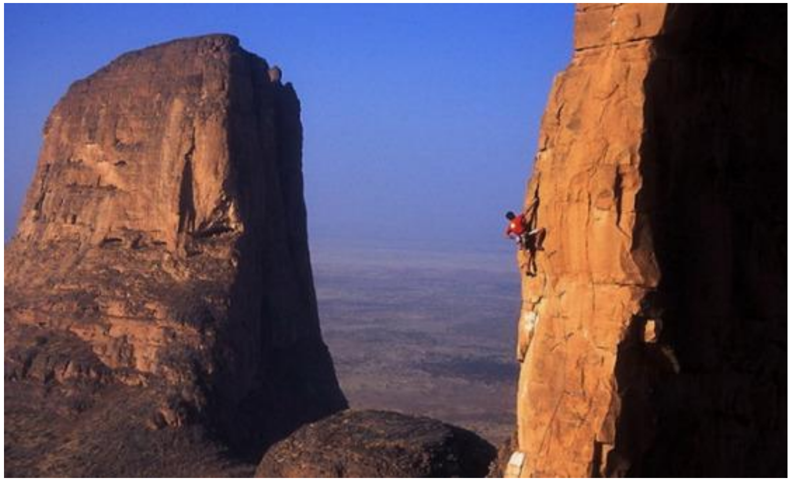




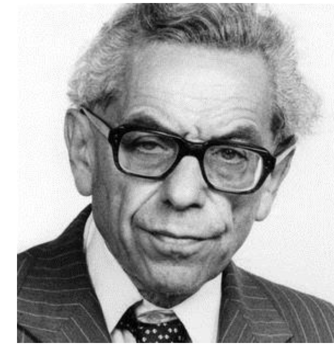
Extremal Problem

- **Erdős.** What is the *minimum* number of *DD* that can be determined by a set of n points in the plane?





A Word from Erdős

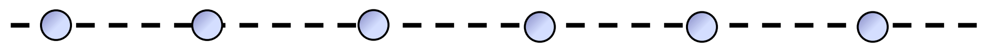


- For the celebrations of his 80'th birthday, Erdős compiled a survey of his favorite contributions to mathematics, in which he wrote

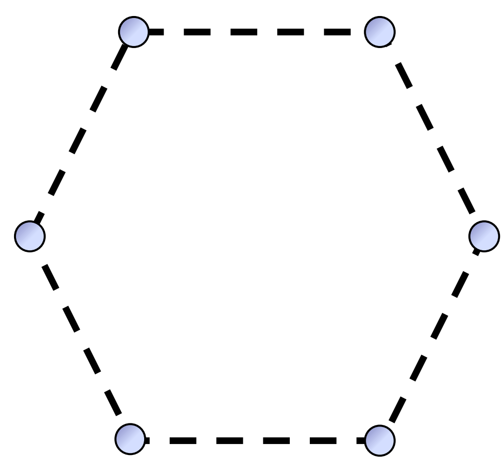
  "My most striking contribution to geometry is, no doubt, my problem on the number of distinct distances."

Simple Upper Bounds

- Evenly spaced on a line: $n - 1$ ***DD***.

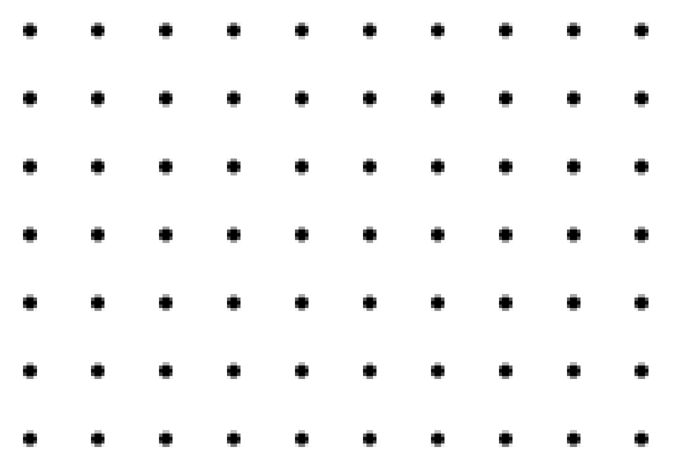


- Regular n-gon: $\left\lfloor \frac{n}{2} \right\rfloor$ ***DD***.



An Improved Upper Bound

- **Erdős ’46**: A $\sqrt{n} \times \sqrt{n}$ integer lattice determines $O\left(\frac{n}{\sqrt{\log n}}\right)$ *DD*.
- **Landau–Ramanujan**: There are $O\left(\frac{n}{\sqrt{\log n}}\right)$ integers of size at most n that can be expressed as the sum of two squares.

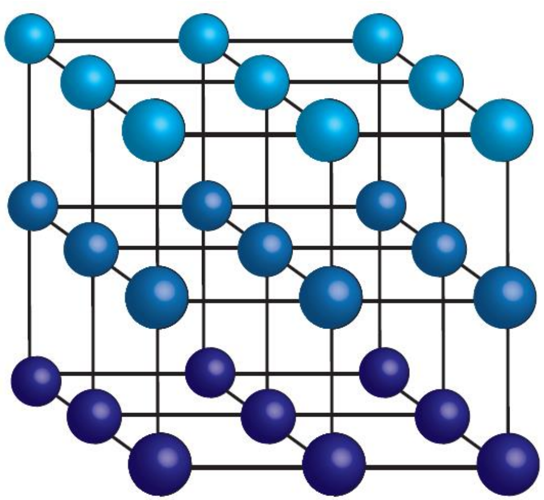


$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Authors + Year	Bound
Erdős `46	$\mathbf{\Omega}(n^{1/2})$
Moser `52	$\mathbf{\Omega}(n^{2/3})$
Chung `84	$\mathbf{\Omega}(n^{5/7})$
Chung, Szemerèdi, and Trotter `92	$\mathbf{\Omega}(n^{4/5}/\log n)$
Szèkely `97	$\mathbf{\Omega}(n^{4/5})$
Solymosi and Tóth `01	$\mathbf{\Omega}(n^{6/7})$
Tardos `01	$\mathbf{\Omega}(n^{0.8634})$
Katz and Tardos `04	$\mathbf{\Omega}(n^{0.8641})$
Guth and Katz `10	$\mathbf{\Omega}(n/\log n)$

Open Variant: Distinct distances in $\mathbb{R}^3$

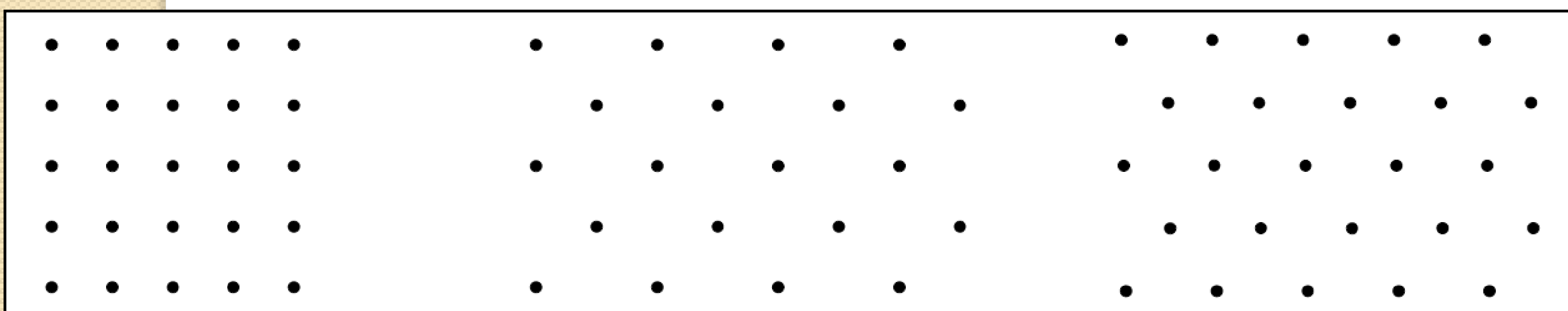
- Upper bound: $O\left(n^{2/3}\right)$.
  - Obtained from $n^{1/3} \times n^{1/3} \times n^{1/3}$ grid.
- Lower bound: $\Omega^*\left(n^{3/5}\right)$.
  - Obtained by combining the results in **Solymosi and Vu `08** and in **Guth and Katz `10**.



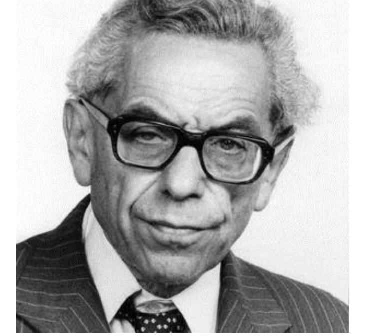
Variant #2: Optimal Configurations

- **Problem.** Characterize the sets of n points that determine $O\left(n/\sqrt{\log n}\right)$ *DD*.

- Some known examples:



Conjectures by Erdős



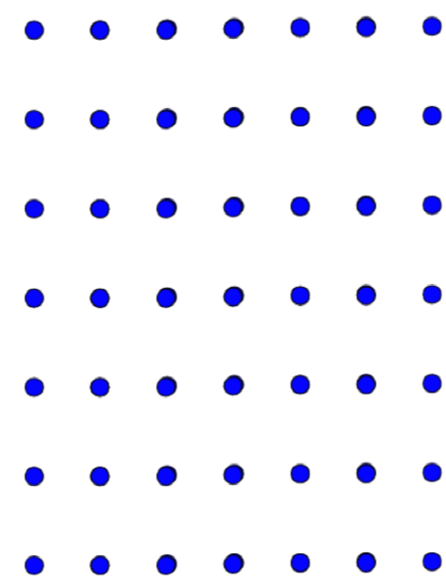
- **Conjecture.** A configuration that determines $O\left(n/\sqrt{\log n}\right)$ *DD* must *have lattice structure*.
  - Every such set can be *covered by a* relatively *small number of lines*.
  - For every such set there exists *a line that contains $\Omega(\sqrt{n})$ points* of the set.
  - A line that contains $\Omega(n^{\varepsilon})$ *points* of the set?
  - **Szemerédi (1975?).** There exists a line that contains $\Omega\left(\sqrt{\log n}\right)$ *points* of the set.

Rectangular Lattices

- For every integer $r > 1$, consider the lattice

$$L_r = \left\{(i, j\sqrt{r} \,\middle|\, i,j \in \mathbb{N} \quad 1 \le i,j \le \sqrt{n}\right\}.$$

- The number of ***DD*** spanned by any L_r is $O\left(n/\sqrt{\log n}\right)$.
- Relies on a generalization of the Landau-Ramanujan result, originally from Bernays' 1912 Ph.D. dissertation, under the supervision of Landau.



What is Known

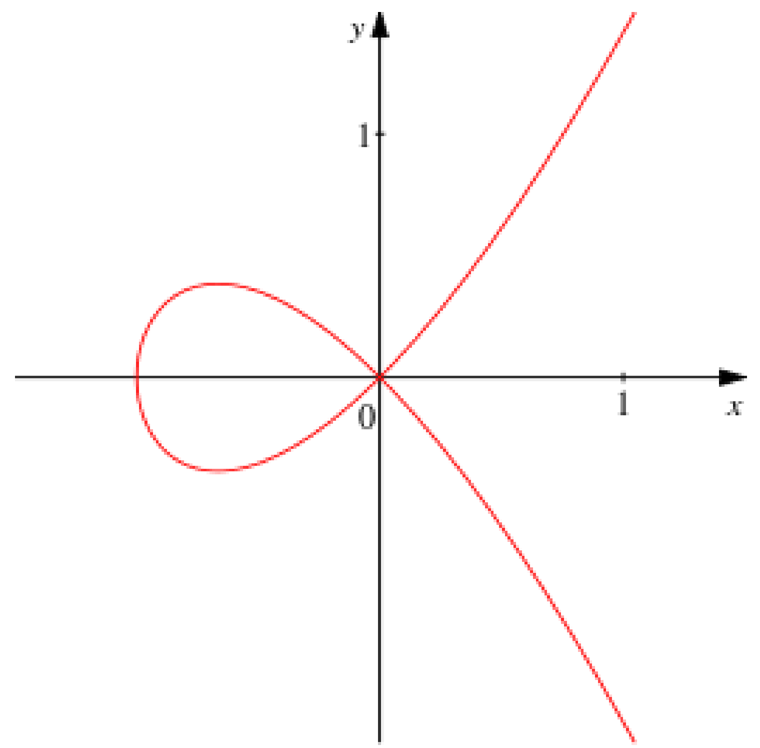
- P – a set of n points spanning $O\left(n/\sqrt{\log n}\right)$ ***DD***.
  - **Szemerédi (1975?).** There exists a line that contains $\Omega\left(\sqrt{\log n}\right)$ points of P (can be improved to $\Omega(\log n)$ using modern tools).
  - **Pach and de Zeeuw `14** and **S', Zahl, and de Zeeuw `14**:
    - No line contains $\Omega\left(n^{7/8}\right)$ points of P.
    - No circle contains $\Omega\left(n^{5/6}\right)$ Points of P.
    - No other irreducible constant-degree polynomial curve contains $\Omega\left(n^{3/4}\right)$ points of P.

New Properties

- **Theorem (Lund, S', de Zeeuw).** Given a set P of n points spanning $O\left(n/\sqrt{\log n}\right)$ ***DD***. For any $k = O\left(n^{1/2}\right)$, at least one of the following holds:
  - The exists a line or a circle containing $\Omega(k)$ points of P.
  - There exist $\Omega\left(\frac{n^{8/5-\varepsilon}}{k^{4/3}}\log^{1/12} n\right)$ lines that contain $\Omega\left(\sqrt{\log n}\right)$ points of P.

Many Collinear Triples

- **Sylvester.** A set of n points on a cubic curve that form a group yield about $\frac{n^2}{6}$ collinear triples.
- No line contains four points of the set.



Lines with k points

- **Solymosi and Stojaković `13.** For any integer $k > 3$, there exists a set P of n points in $\mathbb{R}^2$ with $\Omega\left(n^{2-c/\sqrt{logn}}\right)$ lines that contain k points of P and no line that contains $k+1$ points of P.
- What happens when k depends on n?

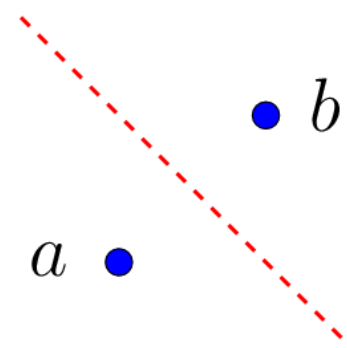
Our Hope

- **Conjecture.** Let P be a set of n points in $\mathbb{R}^2$. If there exist $\Omega\left(n^{8/5-\varepsilon}\right)$ lines that contain $\Omega\left(\sqrt{\log n}\right)$ points of P, then there exists a constant-degree (cubic?) curve that contains n^{β} points of P.

Bisector Energy

- P – a set of n points in $\mathbb{R}^2$.
- For any $a, b \in P$, we denote by $\boldsymbol{B}_{ab}$ the perpendicular bisector of a and b.
- The *bisector energy* of P is the cardinality of the set

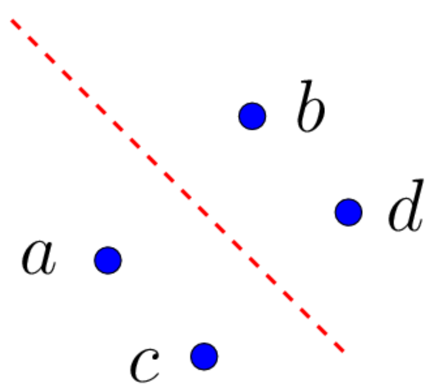
$$\boldsymbol{B}(P) = \{(a, b, c, d) \in P^4 \mid \boldsymbol{B}_{ab} = \boldsymbol{B}_{cd}\}.$$



Trivial Energy Bounds

$$\boldsymbol{B}(P) = \{(a, b, c, d) \in P^4 \mid \boldsymbol{B}_{ab} = \boldsymbol{B}_{cd}\}.$$

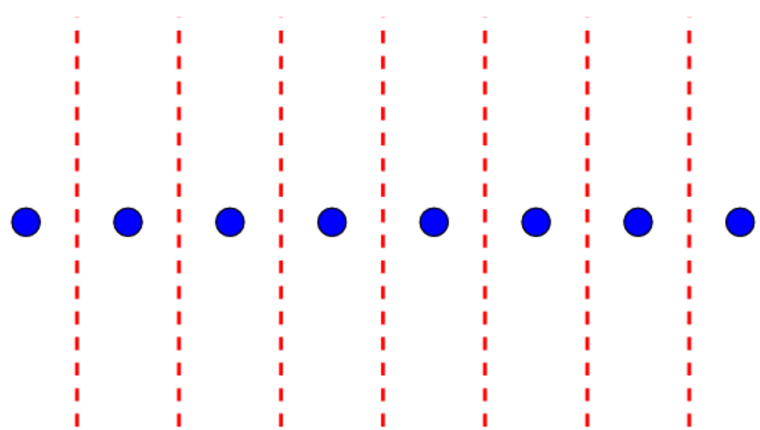
- What is a trivial upper bound on $|\boldsymbol{B}(P)|$?
  - For any choice of a, b, c, there is at most one valid choice for d.
  - Thus, $|\boldsymbol{B}(P)| = O(n^3)$.

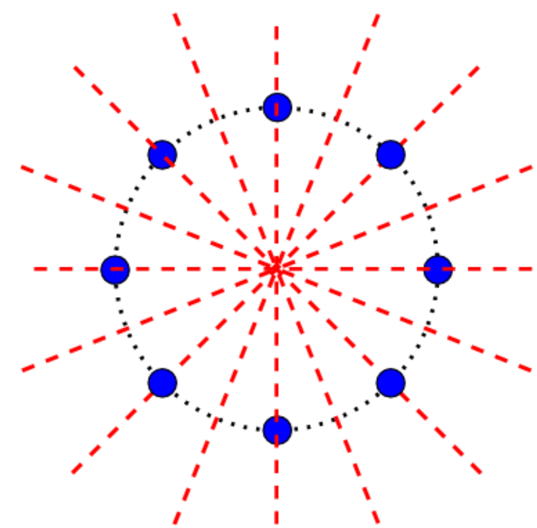


Trivial Energy Bounds

$$\boldsymbol{B}(P) = \{(a, b, c, d) \in P^4 \mid \boldsymbol{B}_{ab} = \boldsymbol{B}_{cd}\}.$$

- What is a trivial *lower* bound on $|\boldsymbol{B}(P)|$?
  - $|\boldsymbol{B}(P)| = \Omega(n^3)$.





Bisector Energy Bound

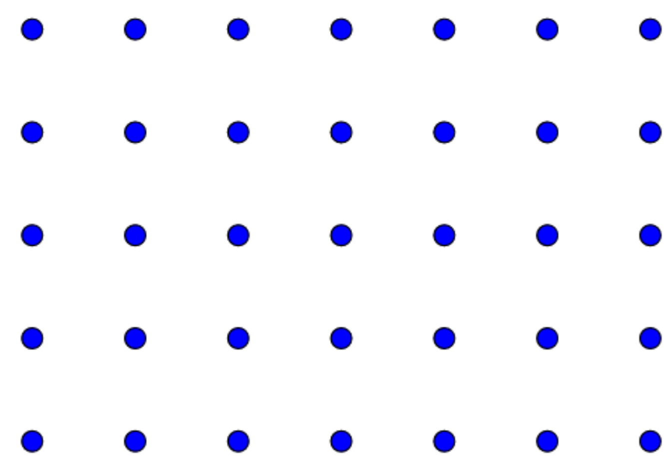
- **Theorem (Lund, S', de Zeeuw).** Let P be a set of n points in $\mathbb{R}^2$, such that every line or circle contains $O(m)$ points of P. Then

$$|\boldsymbol{B}(P)| = O\left(m^{2/5}n^{12/5+\varepsilon} + mn^2\right).$$

- **Conjecture.** The correct bound is $|\boldsymbol{B}(P)| = O^*(mn^2)$.
  - Our bound matches this when $m = \Omega\left(n^{2/3+\varepsilon}\right)$.
  - Matching lower bound for any m.

Bisector Energy: Lower Bound

- Every line or circle contains $O(m)$ points of P.
- We wish to prove $|\boldsymbol{B}(P)| = \Omega^*(mn^2)$.
- When $m = \Omega\left(n^{1/2}\right)$, we can take an $m \times (n/m)$ integer lattice.

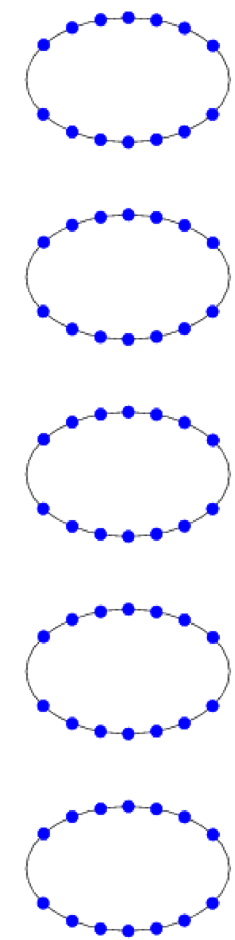


Bisector Energy: Lower Bound (2)

- Every line or circle contains $O(m)$ points of P.
- We wish to prove

$$|\boldsymbol{B}(P)| = \Omega(mn^2).$$

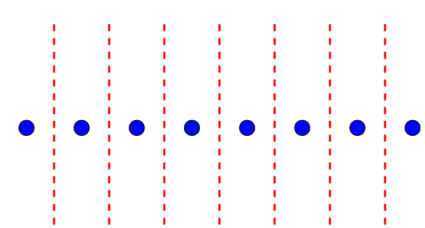
- For any m:
  - $m/2$ ellipses, evenly spaced above each other.
  - Every ellipse contains $2n/m$ points, with reflection symmetry around its horizontal axis.

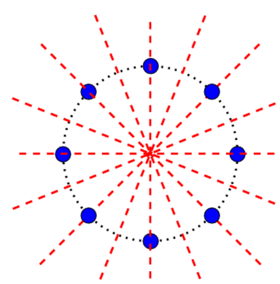


Distinct Bisectors

- $DB(P)$ – the number of distinct bisectors spanned by pairs of points of P.
- **Corollary.** Let P be a set of n points in $\mathbb{R}^2$, such that every line or circle contains $O(m)$ points of P. Then

$$DB(P) = \Omega\left(\min\left\{\frac{n^{8/5-\varepsilon}}{m^{2/5}}, \frac{n^2}{m}\right\}\right).$$





Distinct Bisectors: Proof Sketch

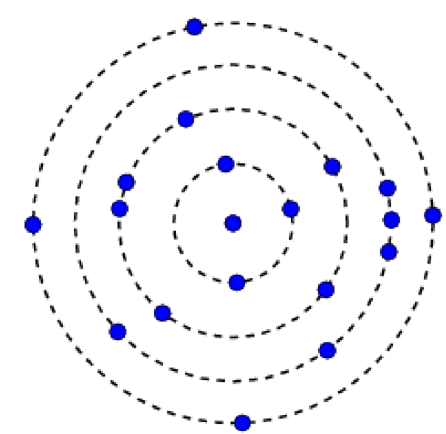
- For a line ℓ, we set
$$E_\ell(P) = \{(a,b) \in P^2 \mid \boldsymbol{B}_{ab} = \ell\}.$$
- By the *Cauchy-Schwartz* inequality
$$|\boldsymbol{B}(P)| = \sum_\ell \binom{|E_\ell(P)|}{2} \geq \frac{\left(\sum_\ell |E_\ell(P)|\right)^2}{DB(P)}$$
$$= \Omega\left(\frac{n^4}{DB(P)}\right).$$
    - The bound is obtained by combining this with the upper bound for $\boldsymbol{B}(P)$.

From Few Distinct Distances to Bisector Energy

- P – a set of n points, such that pairs of points span $O\left(n/\sqrt{\log n}\right)$ ***DD***.

$$T = \{(a,b,c) \in P^3 \mid |ab| = |ac|\}.$$

  - For any $a \in P$, the points of $P \setminus \{p\}$ are contained in $O\left(n/\sqrt{\log n}\right)$ circles centered at a.
  - $P_{a,i}$ – the set of points of P on the i'th circle around a.



Counting Triples

$$T = \{(a,b,c) \in P^3 \mid |ab| = |ac|\}.$$

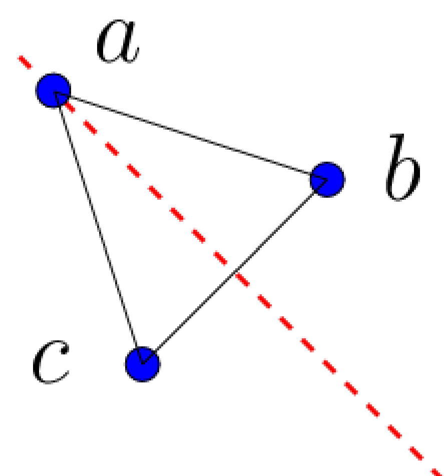
- $P_{a,i}$ – the set of points of P on the i'th circle around a. Notice that $\sum_i |P_{a,i}| = n - 1$.
- By the *Cauchy-Schwarz* inequality

$$|T| = \sum_{a \in P} \sum_i \binom{|P_{a,i}|}{2} = n \cdot \Omega\left(\frac{n^2}{n/\sqrt{\log n}}\right)$$
$$= \Omega\left(n^2\sqrt{\log n}\right).$$

Counting Triples Again

$$T = \{(a, b, c) \in P^3 \mid |ab| = |ac|\}.$$

- A triple $(a, b, c) \in P^3$ is in T iff $\boldsymbol{B}_{bc}$ is incident to a.
- $|T|$ is the number of incidences between P and a multiset of $\binom{n}{2}$ lines.
- By our lower bound for $|T|$, the number of incidences is $\Omega\left(n^2\sqrt{\log n}\right)$.
  - *How is this possible?*



Taking Multiplicities into Account

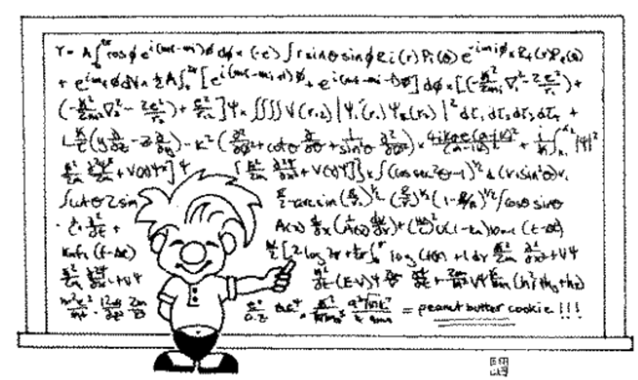
- Every line of "multiplicity" at least k contributes $\Omega(k)$ to the energy.
- By the bound on the energy, the number of such lines is

$$O\left(\frac{m^{2/5}n^{12/5+\varepsilon}+mn^2}{k}\right)$$

- Since no line contains $> m$ points of P, lines of multiplicity $\geq k = \Theta\left(m^{7/5}n^{2/5+\varepsilon}\right)$ cannot yield $\Omega\left(n^2\sqrt{\log n}\right)$ incidences.

Lines with a Low Multiplicity

- There are $\Omega\left(n^2\sqrt{\log n}\right)$ incidences between the point set P and a multiset of lines with multiplicities $O\left(m^{7/5}n^{2/5+\varepsilon}\right)$.
- Lines with $o\left(\sqrt{\log n}\right)$ points also cannot yield $\Omega\left(n^2\sqrt{\log n}\right)$ incidences.
- A *straightforward* analysis shows that the number of remaining lines is $\Omega\left(m^{-7/5}n^{8/5-\varepsilon}\right)$.



Bounding the Bisector Energy

$$\boldsymbol{B}(P) = \{(a,b,c,d) \in P^4 \mid \boldsymbol{B}_{ab} = \boldsymbol{B}_{cd}\}.$$

- A quadruple $(a,b,c,d) \in P^4$ is in $\boldsymbol{B}(P)$ iff

$$(a_x - b_x)(c_y - d_y) = (c_x - d_x)(a_y - b_y),$$

and

$$(a_y - b_y)(a_y + b_y - c_y - d_y) = (a_x - b_x)(c_x + d_x - a_x - b_x).$$

Incidences in $\mathbb{R}^4$

- We consider quadruples $(a, b, c, d) \in P^4$.
- For every pair (b, d), define a point in $\mathbb{R}^4$.
- For every pair (a, c), define a two-dimensional surface in $\mathbb{R}^4$, defined by

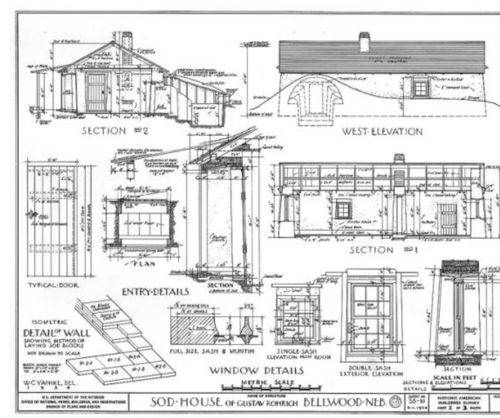
$$(a_x - z_1)(c_y - z_4) = (c_x - z_3)(a_y - z_2),$$

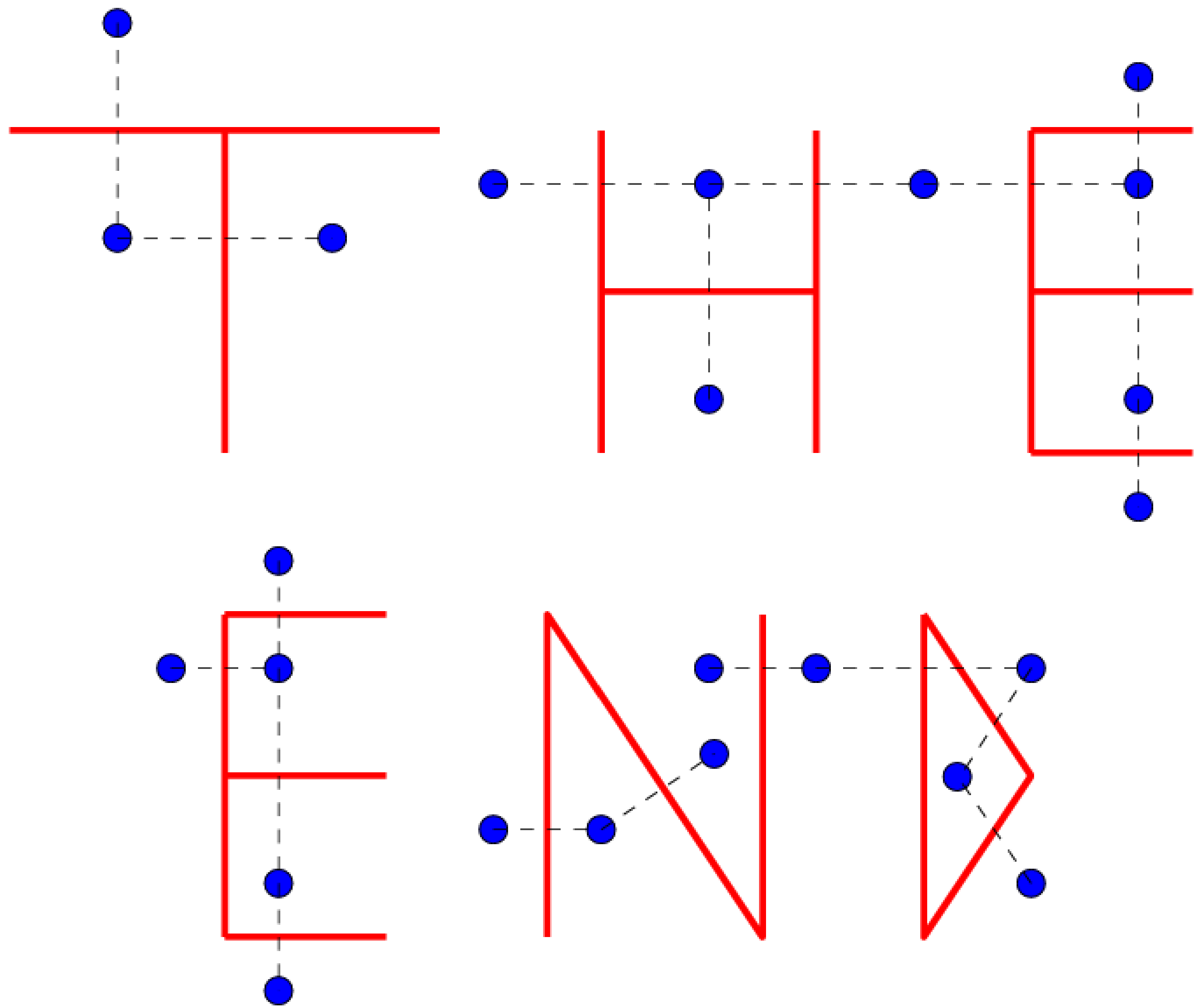
and

$$(a_y - z_2)(a_y + z_2 - c_y - z_4) = (a_x - z_1)(c_x + z_3 - a_x - z_1).$$

Solving the Incidence Problem

- Some high level details:
  - We show that the incidence graph contains no copy of $K_{2,m}$.
  - We show that the incidence graph can be partitioned to many connected components with no edges between them.

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