A Short Proof That χ Can be Bounded ϵ Away from $\Delta + 1$ toward ω*

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Abstract: In 1998 the second author proved that there is an $\epsilon > 0$ such that every graph satisfies $\chi \leq \lceil (1-\epsilon)(\Delta+1) + \epsilon\omega \rceil$. The first author recently proved that any graph satisfying $\omega > \frac{2}{3}(\Delta + 1)$ contains a stable set intersecting every maximum clique. In this note, we exploit the latter result to give a much shorter, simpler proof of the former. Working from first principles, we omit only some five pages of proofs of known intermediate results (which appear in an extended version of this paper), and the proofs of Hall's Theorem, Brooks' Theorem, the Lovász Local Lemma, and Talagrand's Inequality. © 2015 Wiley Periodicals, Inc. J. Graph Theory 81: 30–34, 2016

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1. INTRODUCTION

Much work has been done toward bounding the chromatic number χ of a graph in terms of the clique number ω and the maximum size of a closed neighborhood $\Delta + 1$, which are trivial lower and upper bounds on the chromatic number, respectively. Recently, much of this work has been done in pursuit of a conjecture of Reed, who proposed that the average of the two should be an upper bound for χ, modulo a round-up:

Conjecture 1 [11]. *Every graph satisfies* $\chi \leq \lceil \frac{1}{2}(\Delta + 1 + \omega) \rceil$.

This conjecture has been proven for some restricted classes of graphs [1, 7, 6, 9], sometimes in the form of a stronger local conjecture posed by King or *superlocal* conjecture posed by Edwards and King [4, 2]; all three forms are known to hold in the fractional relaxation [2].

For general graphs, we only know that we can bound the chromatic number by some nontrivial convex combination of ω and $\Delta + 1$:

Theorem 2 [11]. *There exists an* $\epsilon > 0$ *such that every graph satisfies*

$$\chi \leq \lceil (1 - \epsilon)(\Delta + 1) + \epsilon\omega \rceil.$$

The original proof of this theorem is quite long and complicated, requiring a careful probabilistic approach applied to a specific structural decomposition. In this note, we give a much shorter, simpler proof that exploits the following new existence condition for a stable set hitting every maximum clique, the proof of which from first principles is itself short and simple:

Theorem 3 [5]. *Every graph satisfying* $\omega > \frac{2}{3}(\Delta + 1)$ *contains a stable set hitting every maximum clique.*

This result is a strengthening of a result of Rabern [10], which could be used to similar effect.

2. A PROOF SKETCH

We sketch the proof here, prove the necessary lemmas in the following two sections, then finally prove the theorem more formally.

Suppose G is a minimum counterexample to Theorem 2 for some fixed ϵ. Applying Theorem 3 and Brooks' Theorem tells us that G satisfies $\omega \leq \frac{2}{3}(\Delta + 1)$ and $\Delta > \frac{1}{\epsilon}$. Our proof then considers two cases: If every neighborhood contains much fewer than $\binom{\Delta}{2}$ edges, we can apply a simple probabilistic argument. Otherwise we have a vertex v whose neighborhood contains almost $\binom{\Delta}{2}$ edges. The fact that $\omega \leq \frac{2}{3}(\Delta + 1)$ tells us that there is a large antimatching in $N(v)$ (i.e. a large matching in the complement of G induced on $N(v)$), and since there are few edges between $N(v)$ and $G - v$, we can take an optimal coloring of $G - N(v) - v$ and extend it to a coloring of G in which many pairs of the antimatching are monochromatic, which is enough to contradict the minimality of G.

3. DEALING WITH SPARSE NEIGHBORHOODS

Theorem 10.5 in [8], which is a straightforward application of Talagrand's Inequality, gives us a bound on the chromatic number when no neighborhood contains almost $\binom{\Delta}{2}$ edges:

Theorem 4. *There is a Δ_0 such that for any graph with maximum degree $\Delta > \Delta_0$ and for any $B > \Delta(\log \Delta)^3$, if no $N(v)$ contains more than $\binom{\Delta}{2} - B$ edges then $\chi(G) \leq (\Delta + 1) - \frac{B}{e^6 \Delta}$.*

We let $\alpha = B/\binom{\Delta}{2}$ and restate this theorem as follows:

Corollary 5. *There is a Δ_0 such that for any graph with maximum degree at most $\Delta > \Delta_0$ and for any $\alpha > 2(\log \Delta)^3/(\Delta - 1)$, if no $N(v)$ contains more than $(1-\alpha)\binom{\Delta}{2}$ edges then*

$$\chi(G) \leq (\Delta + 1) - \frac{\alpha(\Delta - 1)}{2e^6} \leq \left(1 - \frac{\alpha}{2e^6}\right)(\Delta + 1) + \frac{\alpha}{2e^6}\omega.$$

This is all we need for the case in which no neighborhood contains almost $\binom{\Delta}{2}$ edges.

4. DEALING WITH DENSE NEIGHBORHOODS

We need the following theorem to extend a coloring when we have a dense neighbourhood.

Theorem 6. *Let α be any positive constant and let ϵ be any constant satisfying $0 < \epsilon < \frac{1}{6} - 2\sqrt{\alpha}$. Let G be a graph with $\omega \leq \frac{2}{3}(\Delta + 1)$ and let v be a vertex whose neighborhood contains more than $(1-\alpha)\binom{\Delta}{2}$ edges. Then*

$$\chi(G) \leq \max\{\chi(G - v), (1 - \epsilon)(\Delta + 1)\}.$$

This immediately implies:

Corollary 7. *Let ρ be a positive constant satisfying $\rho \leq \frac{1}{160}$, let G be a graph with maximum degree at most Δ, $\omega \leq \frac{2}{3}(\Delta + 1)$, and let v be a vertex whose neighborhood contains at least $(1-\rho)\binom{\Delta}{2}$ edges. Then*

$$\chi(G) \leq \max\{\chi(G - v), (1 - \rho)(\Delta + 1)\}.$$

Before we prove Theorem 6 we need to lay out one more simple fact:

Lemma 8. *Every graph G contains an antimatching of size $\lfloor \frac{1}{2}(n - \omega(G)) \rfloor$.*

Proof. Let M be a maximum antimatching; there are $n - 2|M|$ vertices outside M, and these vertices must form a clique. Thus $\omega(G) \geq n - 2|M|$; the result follows.

Proof of Theorem 6. We may assume that $d(v) = \Delta$ since if this is not the case we can hang pendant vertices from v, and we may assume $\alpha < \frac{1}{144}$, since otherwise no valid value of ϵ exists. Our approach is as follows. We first partition the closed neighborhood of v, denoted $\tilde{N}(v)$, into sets D_1, D_2, and D_3 such that D_1 and D_2 are small, each $u \in D_2$ has few neighbors outside $D_2 \cup D_3$, and each $u \in D_3$ has very few neighbors outside D_3. In particular, $v \in D_3$. Then, using at most $\max\{\chi(G - v), (1 - \epsilon)(\Delta + 1)\}$ colors, we first color $G - (D_2 \cup D_3)$, then greedily extend the coloring to D_2. Finally, we exploit the existence of a large antimatching in $G|D_3$ and extend the coloring to D_3 using an elementary result on list colorings.

Since $N(v)$ contains more than $(1-\alpha)\binom{\Delta}{2}$ edges, the number of edges between $G - \tilde{N}(v)$ and $\tilde{N}(v)$ is less than $\Delta(\Delta-1) - 2(1-\alpha)\binom{\Delta}{2} = \alpha(\Delta^2 - \Delta)$. We set $c_1 = \frac{1}{2}$ and $c_2 = \sqrt{\alpha}$. We partition $N(v)$ into D_1, D_2, and D_3 as follows:

$$D_1 = \{\, u \in \tilde{N}(v) \mid u \text{ has more than } c_1(\Delta+1) \text{ neighbors outside } \tilde{N}(v) \,\}$$
$$D_2 = \{\, u \in \tilde{N}(v) \setminus D_1 \mid u \text{ has more than } c_2(\Delta+1) \text{ neighbors outside } \tilde{N}(v) \setminus D_1 \,\}$$
$$D_3 = \tilde{N}(v) \setminus (D_1 \cup D_2)$$

Let β_1 denote $|D_1|/(\Delta+1)$ and let β_2 denote $|D_2|/(\Delta+1)$. Thus $|D_3| = (1-\beta_1-\beta_2)(\Delta+1)$. Since there are fewer than $\alpha\Delta^2$ edges between $\tilde{N}(v)$ and $G - \tilde{N}(v)$, we can see that $|D_1| < \alpha\Delta^2/(c_1(\Delta+1)) < 2\alpha\Delta$. Further, since $\alpha < \frac{1}{144}$, we have $2\alpha\Delta < \frac{1}{6}\sqrt{\alpha}(\Delta+1)$. Note that every vertex in D_1 has more neighbors outside $\tilde{N}(v)$ than in $\tilde{N}(v)$, so there are fewer than $\alpha\Delta^2$ edges between $D_2 \cup D_3$ and $G - (D_2 \cup D_3)$. Thus $|D_2| < \alpha\Delta^2/(c_2(\Delta+1)) < \sqrt{\alpha}(\Delta+1)$. Therefore $\beta_1 < 2\alpha < \frac{1}{6}\sqrt{\alpha}$ and $\beta_2 < c_2 = \sqrt{\alpha}$. By the first of these two facts, we can see that v is in D_3.

Now let k denote $\lfloor (1-\epsilon)(\Delta+1) \rfloor$, let k' denote $\max\{k, \chi(G-v)\}$, and take a k'-coloring of $G - (D_2 \cup D_3)$. We greedily extend this to a k'-coloring of $G - D_3$. To see that this is possible, note that while extending, every vertex in D_2 has at most $|D_1| + |D_2| + c_1(\Delta+1) - 1 = (\beta_1 + \beta_2 + c_1)(\Delta+1) - 1$ colored neighbors, so each vertex has at least $k - (\beta_1+\beta_2+c_1)(\Delta+1) + 1 > (\frac{1}{2} - \epsilon - \frac{7}{6}\sqrt{\alpha})(\Delta+1) > 0$ available colors, so we can indeed extend to all vertices of D_2 greedily.

Extending the partial coloring to D_3 takes a little more finesse. Let M be a maximum antimatching in $G|D_3$. We now define the graph G_3 as a clique of size $|D_3|$ minus $|M|$ vertex-disjoint edges. Note that $G|D_3$ is a subgraph of G_3. By assumption, $\omega(G|D_3) \le \frac{2}{3}(\Delta+1)$. Solving for $|M|$ in Lemma 8 gives $|M| \ge \frac{1}{2}|D_3| - \frac{1}{3}(\Delta+1)$. A classical result of Erdős, Rubin, and Taylor on list colorings states that if H is a complete multipartite graph with t parts, each of size at most 2, then $\chi_l(H) = \chi(H) = t$ [3] (this can be proven easily using induction and Hall's Theorem). Combining this result with the bound on $|M|$ from Lemma 8 tells us that $\chi_l(G|D_3) \le \chi_l(G_3) = \chi(G_3) = |D_3| - |M| \le \frac{1}{2}|D_3| + \frac{1}{3}(\Delta+1) \le \frac{5}{6}(\Delta+1)$. It follows that if we give each vertex of D_3 a list of at least $\frac{5}{6}(\Delta+1)$ colors, we can find a coloring of $G|D_3$ such that every vertex gets a color from its list.

We extend the partial coloring of $G - D_3$ to a coloring of G by assigning each vertex u in D_3 a list ℓ_u consisting of all colors from 1 to k not appearing in $N(u) \setminus D_3$. Each list has size at least $k - (\beta_2 + c_2)(\Delta+1) > (1 - \epsilon - \beta_2 - c_2)(\Delta+1) - 1 > (1 - \epsilon - 2\sqrt{\alpha})(\Delta+1) - 1 > \frac{5}{6}(\Delta+1) - 1$ (the last inequality holds since, by hypothesis, $\epsilon < \frac{1}{6} - 2\sqrt{\alpha}$). Since the list sizes are integers, each list has size at least $|D_3| - |M|$. Therefore we can extend the k'-coloring of $G - D_3$ to a k'-coloring of G. This completes the proof.

5. PUTTING IT TOGETHER

We can now prove Theorem 2.

Proof of Theorem 2. Take Δ_0 from the statement of Corollary 5 and set ϵ as $\min\{\frac{1}{\Delta_0}, \frac{1}{320e^6}\}$.

Let G be a counterexample on a minimum number of vertices and denote its maximum degree and clique number by Δ and ω respectively. If $\Delta \leq \Delta_0$ then the result is implied by Brooks' Theorem, so we can assume $\Delta > \Delta_0$. If $\omega < \frac{2}{3}(\Delta + 1)$, then Theorem 3 guarantees that we have a maximal stable set S such that $\Delta(G - S) < \Delta$ and $\omega(G - S) < \omega$. By the minimality of G we have a proper coloring of $G - S$ using

$$\lceil (1-\epsilon)(\Delta(G-S)+1) + \epsilon\omega(G-S) \rceil < \lceil (1-\epsilon)(\Delta+1) + \epsilon\omega \rceil$$

colors, to which we can add S as a color class, giving the desired coloring of G. So G satisfies $\omega \leq \frac{2}{3}(\Delta + 1)$.

Now G must be vertex-critical, must satisfy $\omega \leq \frac{2}{3}(\Delta + 1)$ and $\Delta > \Delta_0$, and must have chromatic number $> (1 - \frac{1}{320e^6})(\Delta + 1)$. Thus by Corollary 7 there is no vertex v such that the neighborhood of v contains more than $(1 - \frac{1}{160})\binom{\Delta}{2}$ edges. The theorem now follows immediately from Corollary 5.

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