Discrete Mathematics 340 (2017) 988–990

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

Note on terminal-pairability in complete grid graphs

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ARTICLE INFO

Article history:
Received 22 June 2016
Received in revised form 6 October 2016
Accepted 14 January 2017

Keywords:
Path-pairability
Terminal-pairability
Complete grid graphs

ABSTRACT

We affirmatively answer and generalize the question of Kubicka, Kubicki and Lehel (1999) concerning the path-pairability of high-dimensional complete grid graphs. As an intriguing by-product of our result we significantly improve the estimate of the necessary maximum degree in path-pairable graphs, a question originally raised and studied by Faudree, Gyárfás, and Lehel (1999).

1. Introduction

We discuss a graph theoretic concept of *terminal-pairability* emerging from a practical networking problem introduced by Csaba, Faudree, Gyárfás, Lehel, and Schelp [1] and further studied by Faudree, Gyárfás, and Lehel [2–4] and by Kubicka, Kubicki and Lehel [7]. Given a simple undirected graph $G = (V(G), E(G))$ and an undirected multigraph $D = (V(D), E(D))$ on the same vertex set $(V(D) = V(G))$ we say that G can realize the edges $e_1, \dots, e_{|E(D)|}$ of D if there exist edge disjoint paths $P_1, \dots, P_{|E(D)|}$ in G such that P_i joins that endpoints of e_i, $i = 1, 2, \dots, |E(D)|$. We call D and its edges the *demand graph* and the *demand edges* of G, respectively. Given G and a family $\mathcal{F}$ of (demand)graphs defined on $V(G)$ we call G *terminal-pairable* with respect to $\mathcal{F}$ if every demand graph in $\mathcal{F}$ can be realized in G. In particular, let $|V(G)|$ be even and let $\mathcal{M}$ consist of all perfect matchings of the complete graph on $|V(G)|$ vertices; we call G a *path-pairable* graph if it is terminal-pairable with respect to $\mathcal{M}$.

A long-standing open question concerning path-pairability of graphs is the minimal possible value of the maximum degree $\Delta(G)$ of a path-pairable graph G. Faudree, Gyárfás, and Lehel [4] proved that the maximum degree has to grow together with the number of vertices in path-pairable graphs. They in fact showed that a path-pairable graph with maximum degree Δ has at most $2\Delta^{\Delta}$ vertices. The result yields a lower bound of order of magnitude $\frac{\log n}{\log\log n}$ on the maximum degree of a path-pairable graph on n vertices. This bound is conjectured to be asymptotically sharp, although to date only constructions of much higher order of magnitude have been found. The best known construction is due to Kubicka, Kubicki, and Lehel [7] who showed that two dimensional complete grids on an even number of vertices (of at least 6) are path-pairable. A two dimensional complete grid is the *Cartesian product* $K_s \square K_t$ of two complete graphs K_s and K_t and it can be constructed by taking the Cartesian product of the sets $\{1, 2, \dots s\}$ and $\{1, 2, \dots t\}$ and joining two vertices if they share a coordinate. Higher dimensional complete grids can be defined similarly; let $n, t_1, \dots, t_n$ be positive integers and let V denote the set of n-dimensional vectors of positive integer coordinates not exceeding t_i in the ith coordinate, that is,

$$V_{(t_1,\dots,t_n)} = \{(a_1, \dots, a_n) : 1 \le a_i \le t_i, i = 1, 2, \dots, n\}.$$

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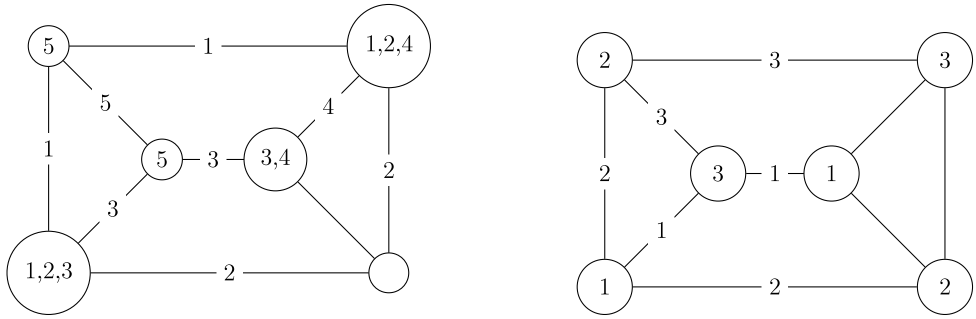
http://dx.doi.org/10.1016/j.disc.2017.01.014



(a) A terminal-pairability problem and its solution.

(b) A path-pairability problem and its solution.

Fig. 1. Examples for terminal-pairability and path-pairability problems. Each vertex is labeled with the demand edges to which it is incident to.

We construct the n-dimensional grid graph $K^n_{(t_1,\ldots,t_n)}$ by taking $V_{(t_1,\ldots,t_n)}$ as its vertex set and we join two vertices by an edge if the corresponding vectors differ at exactly one coordinate. Note that this graph is isomorphic to the Cartesian product $K_{t_1}\square K_{t_2}\square\ldots\square K_{t_n}$. For $t_1 = t_2 = \cdots = t_n = t$ we use the notation $K^n_t = \overbrace{K_t\square\ldots\square K_t}^{n}$. For a more detailed introduction of the Cartesian product of graphs we refer the reader to [6].

With $s = t$ the construction of Kubicka, Kubicki, and Lehel gives examples of path-pairable graphs on $n = s\cdot t$ vertices with maximum degree $2\cdot\sqrt{n}$. This bound was recently improved to $\sqrt{n}$ by Mészáros [8]. It was also conjectured in [7] that $K_t\square K_t\square K_t$ is path-pairable for sufficiently large even values of t.

In this paper we significantly improve the upper bound on the minimal value of Δ by proving path-pairability of high dimensional complete grids. We eventually study the more general terminal-pairability variant of the above path-pairability problem (see Fig. 1) and prove the following theorem:

Theorem 1. *Let $G = K^n_t$ and let $D = (V(D), E(D))$ be a demand graph with $V(D) = V(K^n_t)$ and $\Delta(D) \le \lfloor \frac{t}{6}\rfloor - 2$ even. Then every demand edge of D can be assigned a path in G joining the same endpoints such that the system of paths is edge-disjoint.*

Theorem 1 immediately implies the following corollary:

Corollary 2. *If $t \ge 24$, K^n_t is path-pairable.*

The above construction provides examples of path-pairable graphs on $N = t^n$ vertices with maximum degree $t\cdot n = \log N\cdot\frac{t}{\log t}$. Observe that t can be chosen to be a constant ($t = 24$) thus we have obtained path-pairable graphs on N vertices with $\Delta \approx 5.2\log N$.

Before the proof of Theorem 1 we set the notation and terminology: for $i = 1,\ldots,t$ let L_i be the subgraph of K^n_t induced by $\{(a_1,\ldots,a_{n-1},i) : 1 \le a_j \le t_j, j = 1,2,\ldots,n-1\}$. We call $L_1,\ldots,L_t$ the *layers* of K^n_t. Similarly, by fixing the first $n-1$ coordinates we get t^{n-1} copies of K_t; we denote these complete subgraphs by $l_1,\ldots,l_{t^{n-1}}$ and refer to them as *columns*.

2. Proof of Theorem 1

Given an edge uv of the demand graph with $u, v \in K^n_t$ we replace uv by a path of three edges uu', $u'v'$, and $v'v$ where $u', v' \in K^n_t$ and u, u' and v, v' lie in the same columns and u', v' share the same layer, that is, $u', v' \in L_i$ for some $i \in \{1,2,\ldots,t\}$. Having done that, we consider the new demand edges defined within the t layers and t^{n-1} columns and break the initial problem into $t^{n-1} + t$ subproblems that we solve inductively. We devote the upcoming sections to the detailed discussion of the above described solution plan.

For the discussion of the base case $n = 1$ as well as for the inductive step we use the following theorem:

Theorem 3 ([5]). *Let $K_t(q)$ be a q-regular demand multigraph of the complete graph K_t. If $q \le 2\lfloor\frac{t}{6}\rfloor - 4$, then K_t is terminal-pairable with respect to $K_t(q)$.*

We mention that instead of using Theorem 3 we could use a weaker version of the theorem with $q \le \frac{t}{4+2\sqrt{3}}$ proved by Csaba et al. in [1]. With every further step of our proof unchanged a result similar to Theorem 1 could be proved with a smaller bound on $\Delta(D)$.

Let q be an even number with $2 \le q \le \lfloor\frac{t}{6}\rfloor - 1$ and let $D = (V(D), E(D))$ be a demand multigraph with $V(D) = K^n_t$ and $\Delta(D) \le q$. Let $E'(D)$ denote the set of demand edges whose endvertices lie in different l_i, l_j columns. We construct an auxiliary graph H with $V(H) = V(K^{n-1}_t)$ and project every edge of $E'(D)$ into H by deleting the last coordinates of the endvertices. It is easy to see that $\Delta(H) \le t\cdot q$. We may assume without loss of generality that D is $t\cdot q$-regular by joining additional pairs of vertices or replacing edges by paths of length two if necessary. We use the well known 2-Factor-Decomposition-Theorem of Petersen [9] to distribute the original demand edges among the layers $L_1, \ldots, L_t$ and define new subproblems on them:

Theorem 4 ([9]). *Let G be a $2k$-regular multigraph. Then $E(G)$ can be decomposed into the union of k edge-disjoint 2-factors.*

Obviously, the graph H satisfies the conditions of Theorem 4 thus $E(H)$ can be partitioned into $\frac{q}{2} \cdot t$ edge-disjoint 2-factors. By arbitrarily grouping the above two factors into $\frac{q}{2}$-tuples we can partition $E(H)$ into t edge disjoint subgraphs $H_1, \ldots, H_t$ with $\Delta(H_i) \le q$.

Assume now that the vertices $u = (\underline{a}, i)$ and $v = (\underline{b}, j)$ $(a, b \in [t]^{n-1})$ are joined by a demand edge belonging to $E'(D)$ (thus $\underline{a} \neq \underline{b}$) and assume that the corresponding edge in H is contained by H_k. We then replace the demand edge uv by the following triple of newly established demand edges: $(\underline{a}, i)(\underline{a}, k)$, $(\underline{a}, k)(\underline{b}, k)$, and $(\underline{b}, k)(\underline{b}, j)$. We claim the following:

(i) For every layer L_j the condition $\Delta(L_j) \le q$ holds.
(ii) For every layer l_j the condition $\Delta(l_j) \le 2q$ holds.

The first statement obviously follows from the partition of $E'(D)$. For the second one observe that a vertex v in l_j initially was incident to q demand edges and at most q additional demand edges have been joined to it (otherwise i) is violated). Notice now that every layer L_j contains an $(n-1)$-dimensional subproblem that can be solved (within the layer) by an inductive hypothesis. Also, every layer l_j contains a subproblem (note that the original demand edges in $E(D) \setminus E'(D)$ are incorporated into these subproblems) that can be solved by Theorem 3. That completes our proof.

3. Conclusions and additional remarks

By using Theorem 3 and the described inductive approach we proved that K_t^n is path-pairable for $t \ge 24$, $n \in \mathbb{Z}^+$. It was conjectured by Faudree, Gyárfás, and Lehel [4] that the result of Theorem 3 is true for $q \le \lfloor \frac{t}{2} \rfloor$. If the conjecture is true it improves the constant 4.3 and decreases the lower bound on t in Corollary 2, yet it does not effect the order of magnitude $\log N$ of Δ.

We mention that one particularly interesting and promising path-pairable candidate (with the same order of magnitude but better constant for Δ) is the n-dimensional hypercube Q_n on 2^n vertices ($\Delta(Q_n) = n$). Observe that hypercubes are special members of the above studied complete grid family as $Q_n = (K_2)^n$. Although it is known that Q_n is not path-pairable for even values of n [2], the question is open for odd dimensional hypercubes if $n \ge 5$ (Q_1 and Q_3 are both path-pairable).

Conjecture 5 ([1]). *The $(2k+1)$-dimensional hypercube Q_{2k+1} is path-pairable for all $k \in \mathbb{N}$.*

Acknowledgments

We would like to heartfully thank Professor Jenő Lehel for drawing our attention to the above discussed problems. The third author in particular wishes to express his gratitude for the intriguing discussions about terminal-pairability problems.

The research of the first author was supported by OTKA grant 116769. The research of the third author was supported by OTKA grant 116769.

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