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Cops and Robbers ordinals of cop-win trees

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ABSTRACT

A relational characterization of cop-win graphs was provided by Nowakowski and Winkler in their seminal paper on the game of Cops and Robbers. As a by-product of that characterization, each cop-win graph is assigned a unique ordinal, which we refer to as a CR-ordinal. For finite graphs, CR-ordinals correspond to the length of the game assuming optimal play, with the cop beginning the game in a least favourable initial position. For infinite graphs, however, the possible values of CR-ordinals have not been considered in the literature until the present work.

We classify the CR-ordinals of cop-win trees as either a finite ordinal, or those of the form $\alpha + \omega$, where α is a limit ordinal. For general infinite cop-win graphs, we provide an example whose CR-ordinal is not of this form. We finish with some problems on characterizing the CR-ordinals in the general case of cop-win graphs.

1. Introduction

In the game of Cops and Robbers, cop-win graphs are those for which one cop has a winning strategy to capture the robber. Nowakowski and Winkler [15] were the first ones to provide a characterization of cop-win graphs that is not restricted to finite graphs. Consider an infinite graph G of order $\aleph_\gamma$ and write $\kappa = \aleph_{\gamma+1}$, where $\gamma \geq 0$ is an ordinal. For a vertex v, let $N[v]$ be its closed neighbourhood. Define the relations $\{\leq_\alpha\}_{\alpha<\kappa}$ on $V(G)$ as follows (it is useful to remember that $\leq_\alpha \subseteq V(G)\times V(G)$).

(i) If $u = v$, then $u \leq_0 v$.
(ii) $u \leq_\alpha v$ if for all $x \in N[u]$, there exists $y \in N[v]$ such that $x \leq_\beta y$ for some $\beta < \alpha$.

Observe that for all $\alpha < \beta$, $\leq_\alpha \subseteq \leq_\beta$. It follows that this tower of relations stabilizes at some ordinal $\rho \leq \kappa$; that is, there is a minimum ρ such that $\leq_\rho = \leq_{\rho+1}$. We refer to ρ as the *CR-ordinal* of G. In the case G is a finite graph of order n, note that $\rho \leq n(n-1)$.

A result of [15] that had often been overlooked is that the graph G is cop-win if and only if the relation $\leq_\kappa$ is trivial (that is, $\leq_\kappa = V(G) \times V(G)$). In this paper, we refer to $\leq_\rho$ as the *capture relation* on cop-win graphs. The capture relation has provided insights into algorithms for recognizing cop-win graphs [14], graphs with higher cop number [9], and for generalized Cops and Robbers games [6]. It is not well-defined for graphs whose cop number is greater than one and that is one reason we still do not have a good characterization of k-cop-win graphs, where $k > 0$.

We say that an ordinal κ is a *CR-ordinal* if it is the CR-ordinal of some cop-win graph. We denote the CR-ordinal of a given graph G by $\rho(G)$. A basic question, therefore, is which ordinals are CR-ordinals?

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For finite graphs, $\rho(G)$ is an integer and is the length of the game assuming that the cop and robber play optimally (that is, the cop plays to minimize the length of the game, while the robber plays to maximize it), *maximized* over all possible starting positions for the cop. A related notion is the *capture time* of G, which is the length of the game assuming optimal play *minimized* over starting positions for the cop; see [4]. In the infinite case, we can no longer directly interpret the CR-ordinal as a length of a game. We emphasize that even in the infinite case, a cop captures a robber in a cop-win graph in finitely many rounds as there is no infinite descending sequence of ordinals.

Finite paths demonstrate that every non-negative integer can be a CR-ordinal. Not surprisingly, the classification of CR-ordinals for infinite cop-win graphs is more complex. Indeed, [5] suggests that the family of cop-win graphs is likely not classifiable as for each infinite cardinal $\aleph_\gamma$ there are the maximum possible number $2^{\aleph_\gamma}$ of non-isomorphic cop-win graphs of order $\aleph_\gamma$.

We provide a characterization of CR-ordinals for cop-win trees in Theorem 3.1. The cop-win trees are precisely the *rayless ones*: trees not containing an infinite path as a subgraph. Theorem 3.1 shows that finite ordinals and the ordinals of the form $\alpha + \omega$, where α is a limit ordinal and ω is the first infinite ordinal, are the CR-ordinals witnessed by cop-win trees. In the final section, we consider general cop-win graphs. We provide a family of cop-win graphs, inspired by the graphs in [16], whose CR-ordinals are not those found from trees. Some open problems on the classification of CR-ordinals are stated at the end of the paper.

All the graphs we consider are simple and undirected. We assume the reader is (or will be) familiar with the basic game of Cops and Robbers as defined in, for example, [8,15]. For additional background on Cops and Robbers and its variants, see the book [8] and the surveys [1–3,13]. For background on graph theory, see [11,18]. We denote the distance between vertices u and v by $d(u, v)$. If u is a vertex and S a set of vertices, then $d(u, S)$ is the minimum distance from u to a vertex in S. Let ON be the proper class of ordinals. We use the property that every ordinal is the set of ordinals preceding it in the well ordering of ON. For instance, ω consists of the set of all finite ordinals, and we use this notation throughout. Hence, $\omega = \mathbb{N} = \{0, 1, 2, \ldots\}$. Recall that a *successor ordinal* is one which contains a maximum element; such ordinals are of the form $\alpha + 1$, where α is some ordinal. A *limit ordinal* is not a successor ordinal; for example, ω is a limit ordinal. Transfinite induction is analogous to usual induction, but considers the cases of both successor and limit ordinals. For further reading on ordinals and cardinals, see [10,17].

2. Capture-time ordinal

Throughout this section, let G be a cop-win graph. Before we state our main result in the next section, it will be useful to use the sequence $\{\leq_\alpha\}_{\alpha \leq \rho(G)}$ to introduce a parameter that provides a simpler means of computing $\rho(G)$. Capture-time [4] is a temporal counterpart to the cop number for a graph, measuring the length of the game assuming optimal play. We now provide an ordinal analogue of capture-time.

For $u, v \in V(G)$, define $\eta(u, v) = \alpha$, where α is the minimum ordinal for which $u \leq_\alpha v$ holds. Note that $\eta(u, v)$ is well-defined as ordinals are well-ordered. If $\eta(u, v)$ is finite, then we may interpret it as the length of time it takes a cop on v to capture a robber on u, assuming both play optimally and the robber moves first. Note that the relation is not necessarily symmetric: $\eta(u, v)$ may be different than $\eta(v, u)$. For an example, see Fig. 1 and its corresponding table of η values.

Define $\eta(v)$ as the minimum ordinal α such that $u \leq_\alpha v$ holds for every $u \in V(G)$. Observe that from the definitions, for any $v \in V(G)$ we have that $\eta(v) = \sup_{u \in V(G)} \eta(u, v)$. Finally, we define

$$\eta(G) = \min_{v \in V(G)} \eta(v).$$

When finite, $\eta(G)$ is precisely the capture time [4] of the cop-win graph G, and, hence, we will call such ordinals the *capture-time ordinals* associated with cop-win graphs.

A crucial observation (which follows from the definitions) is that

$$\rho(G) = \sup_{v \in V(G)} \eta(v).$$

In particular, in the finite case, $\rho(G)$ is the maximum capture time over all initial positions of the cop.

Define $\theta(G)$ to be the set of vertices that realize $\eta(G)$; namely

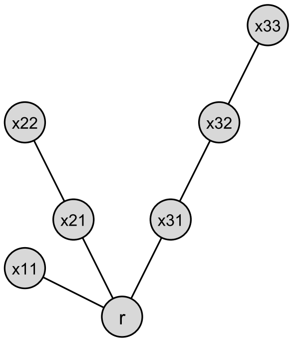
$$\theta(G) = \{v \in V(G) : \eta(v) = \eta(G)\}.$$

Note that by the definitions, $\theta(G) \neq \emptyset$. We may view the set $\theta(G)$ as the set of vertices which are optimal starting positions for the cop. For example, in a finite tree T, $\theta(T)$ is the centre of the tree.

As we have just introduced a number of graph parameters, we give an example that illustrates them. Consider the tree T depicted in Fig. 1, along with its table of η-values. Note that by considering the table, we derive that $\rho(T) = 5$, $\eta(T) = 3$, and $\theta(T) = \{r, x_{31}\}$.

To further explore the capture-time ordinals, we consider the following example. Define the tree $T_\omega = (V(T_\omega), E(T_\omega))$ by setting

$$V(T_\omega) = \{r\} \cup \{x_{i,j} : 0 < i, j < \omega, j \leq i\} \quad \text{and}$$
$$E(T_\omega) = \Big\{\{r, x_{i,1}\} : 0 < i < \omega\Big\} \cup \Big\{\{x_{i,j}, x_{i,j+1}\} : 0 < i, j < \omega, j < i\Big\}.$$



$\eta(\cdot,\cdot)$	r	x_{11}	x_{21}	x_{22}	x_{31}	x_{32}	x_{33}
r	0	4	4	5	3	4	5
x_{11}	1	0	2	5	2	4	5
x_{21}	2	3	0	5	3	4	5
x_{22}	2	3	1	0	3	4	5
x_{31}	3	4	4	5	0	4	5
x_{32}	3	4	4	5	2	0	5
x_{33}	3	4	4	5	2	1	0

Fig. 1. The tree T and its table of η values.

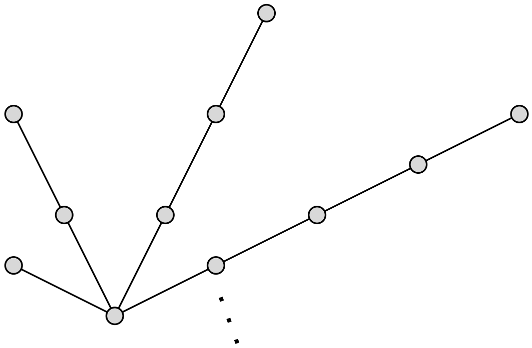


Fig. 2. The tree T_ω.

In other words, for each positive integer n attach a path of length n to the root vertex r. The tree T_ω is suggested in Fig. 2. The tree T_ω possesses no ray and so is cop-win. However, the robber may choose as starting position any given end-vertex, making the capture-time unbounded. We now make this precise in the setting of CR-ordinals. Note that

$$\eta(x_{i(j+k)}, x_{ij}) = i - j \text{ for any } 0 < i, j, k < \omega \text{ such that } j + k \leq i.$$

Hence, we have that

$$\eta(x_{ij}, r) = i \text{ for any } 0 < i, j < \omega, j \leq i.$$

For any $v \in V(T_\omega)$ there exists an $i < \omega$ such that $v \leq_i r$. Further, for any $i < \omega$ there exists a $v \in V(T_\omega)$ such that $v \leq_i r$ does not hold. Thus, we have that $\eta(r) = \omega$.

It is straightforward to derive that for any $0 < i < \omega$, $\eta(r, x_{i1}) = \omega$. Further, for any vertex $v \in V(T_\omega)$, $v \leq_\omega x_{i1}$. Actually, regardless of the robber's moves, the cop can move to r. Hence, for any $0 < i < \omega$ we have that $\eta(x_{i1}) = \omega$.

In addition, for $0 < i, j < \omega, j \leq i$, $\eta(r, x_{ij}) = \omega + j - 1$. Note that for any vertex $v \in V(T_\omega)$ we have $v \leq_{\omega+j-1} x_{ij}$. In addition, for some v, $\eta(v, x_{ij})$ may be smaller than $\omega + j - 1$. Hence, $\eta(x_{ij}) = \omega + j - 1$.

By the above observations, we find that for all positive integers t, there are vertices x with $\eta(x) = \omega + t$. Hence, we have the perhaps surprising conclusion that $\rho(T_\omega) = \omega + \omega = \omega \cdot 2$. Note that $\eta(T_\omega) = \omega$, while $\theta(T_\omega) = \{r\} \cup \{x_{i1} : 0 < i < \omega\}$.

An analogous argument works for any infinite tree obtained by attaching any cardinal number of finite paths to a common root, provided the path lengths are unbounded. Such trees may be uncountable. This gives examples of uncountable graphs with $\eta(G) = \omega \cdot 2$ (see also the construction in the next section).

3. The classification of CR-ordinals for trees

The main result of the paper is the following theorem. Define

$$\Lambda_T = \{\rho(T) : T \text{ is a cop-win tree}\}.$$

Theorem 3.1. *A CR-ordinal for a cop-win tree is either a finite ordinal, or of the form $\alpha + \omega$, where α is a limit ordinal. In particular,*

$$\Lambda_T = \omega \cup \{\alpha + \omega : \alpha \text{ is a limit ordinal}\}.$$

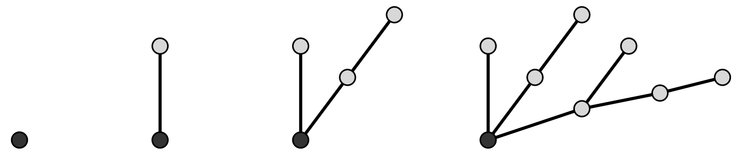


Fig. 3. The rooted trees S_1, S_2, S_3 and S_4, with roots coloured black.

The proof of Theorem 3.1 is divided into two parts: necessity (see Lemmas 3.2 and 3.3) and sufficiency (see Lemma 3.4). For necessity, the main idea of the proof is to determine the relationship between the values of $\eta(T)$, $\rho(T)$ and the radius of the given tree. For sufficiency, we give a transfinite construction of trees whose CR-ordinals have values in Λ_T.

We first present a lemma which is essentially a part of folklore. We omit the proof as it is straightforward.

Lemma 3.2. *Let T be a tree with finite radius. Then $\eta(T)$ equals the radius of T, and $\rho(T)$ equals the diameter of T. In particular, $\rho(T)$ is a finite ordinal.*

The next lemma complements the above for the necessity part of Theorem 3.1.

Lemma 3.3. *If T is a cop-win tree with infinite radius, then $\eta(T)$ is infinite and $\rho(T) = \eta(T) + \omega$.*

In the proof we heavily use that in a tree, an optimal cop strategy is always to go directly towards the robber using the unique geodesic connecting the two players' vertices. Observe that this greedy strategy of the cops may not be optimal for other graph classes. See the results on the game of Zombies and Survivors [7,12].

Proof. As $\eta(u, v) \geq d(u, v)$ for any two vertices $u, v \in V(T)$, we derive immediately that $\eta(T)$ is infinite.

We claim that the set $\theta(T)$ induces a subtree of T with diameter at most two. Now, if $\theta(T)$ contains more than one vertex, let u and v be distinct vertices in $\theta(T)$. If u and v are not adjacent, then let w be any vertex on the unique uv-path. Note that vertex w has the property that for any vertex $x \in V(T)$ at least one of ux-path or vx-path contains vertex w. Therefore, vertex w is also contained in the set $\theta(T)$. Hence, the subgraph induced by $\theta(T)$ in T is connected.

By rechoosing u and v if necessary, suppose for a contradiction that the length of the uv-path is three; say the path is $uu'v'v$. Let T_u, T_v be the two disjoint subtrees obtained from T by removing edge $\{u', v'\}$, such that $u \in V(T_u)$ and $v \in V(T_v)$, respectively. Notice that $u' \in \theta(T)$ implies that either $\eta(v', u') = \eta(T)$, or for each ordinal $\alpha < \eta(T)$, there exists a vertex $w \in V(T_u)$ such that $\eta(w, u') > \alpha$. The second case, however, implies that $\eta(u', v') \geq \eta(T)$. As $v' \in \theta(T)$ as well, we have that $\eta(v', u') = \eta(T)$ or $\eta(u', v') = \eta(T)$. But then we have that

$$\eta(u) \geq \eta(v', u) > \eta(T) \text{ or } \eta(v) \geq \eta(u', v) > \eta(T),$$

which contradicts that both u and v are contained in the set $\theta(T)$. Hence, the claim concerning $\theta(T)$ follows.

By induction we have that for any vertex $v \in V(T) \setminus \theta(T)$, we have that

$$\eta(v) = \eta(T) + d(v, \theta(T)).$$

As the radius of the tree is infinite while the diameter of the subgraph induced by $\theta(T)$ is finite, for any $n < \omega$ there exists some vertex $v \in V(T)$ such that $d(v, \theta(T)) > n$. Hence, we derive that $\rho(T) = \eta(T) + \omega$. $\square$

By elementary ordinal arithmetic, we may assume that η in the sum $\eta + \omega$ is a limit ordinal. This follows since every successor ordinal is of the form $\alpha + k$, where α is a limit ordinal, and k is a finite ordinal.

To prove Theorem 3.1, it is enough to construct a family of trees with CR-ordinals taking all values in the set $\omega \cup \{\alpha + \omega : \alpha \text{ is a limit ordinal}\}$. Finding examples of trees with CR-ordinals equalling all the finite, non-zero ordinals is straightforward. For this, consider the family of finite paths $\{P_n\}_{n\geq 2}$: Lemma 3.2 implies that $\eta(P_n) = \lceil \frac{n-1}{2} \rceil$ and $\rho(P_n) = n - 1$.

We now turn to our construction in the infinite case. We construct a family $\{T_\alpha : \alpha \in \mathrm{ON}\}$, such that for any $\alpha \in \mathrm{ON}$ we have that $\eta(T_\alpha) = \alpha$. This construction, in light of Lemma 3.3, will complete the proof of Theorem 3.1.

The construction is based on the operation of *summing rooted trees*. The basic idea is to form a new root, then append trees to the root by new edges. To be precise, suppose that $\{(T_i, r_i) : i \in \alpha\}$ is a set of disjoint rooted trees indexed by the ordinal α. Form the rooted tree $\bigoplus_{i\in\alpha}(T_i, r_i)$ by adding a new vertex r that is joined to each of the r_i.

We construct our examples by transfinite induction. Let $S_1 = (K_1, r_1)$ with r_1 equalling the single vertex. For any ordinal $\alpha > 1$, assume that all the rooted trees (S_α, r_α) are defined. Let S_α be the rooted tree $\bigoplus_{i<\alpha}(S_i, r_i)$, whose root we denote r_α.

See Fig. 3 for the first four trees in the family $\{(S_\alpha, r_\alpha) : \alpha \in \mathrm{ON}\}$. For simplicity, we refer to these as S_α.

As an aside, by Lemma 3.2, we have that for $0 < n < \omega$, $\eta(S_{n+1}) = n$ and $\rho(S_{n+1}) = 2n - 1$.

Lemma 3.4. *For $\alpha \in \mathrm{ON}$ we have that $\eta(S_{\alpha+1}) = \alpha$.*

Note that taking $T_\alpha = S_{\alpha+1}$ for $\alpha \in \mathrm{ON} \setminus \omega$ constructs the desired family.

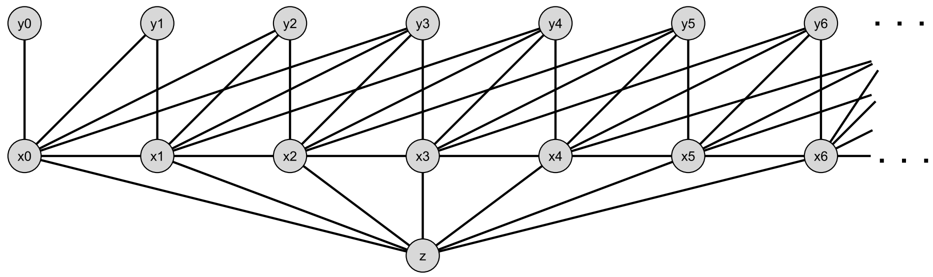


Fig. 4. The Polat graph.

Proof. We prove a slightly stronger statement; namely that for any $\alpha \in \mathrm{ON}$ there exists a vertex $v \in V(S_{\alpha+1})$ such that

$$\eta(S_{\alpha+1}) = \eta(r_{\alpha+1}) = \eta(v, r) = \alpha$$

and, when α is a limit ordinal, there is $\eta(S_\alpha) = \eta(r_\alpha) = \alpha$.

We proceed by transfinite induction on α, with case $\alpha = 1$ being trivially true. Assume that for some $\alpha \in \mathrm{ON}$ and for all $\beta < \alpha$ one has $\eta(S_{\beta+1}) = \beta$, and $\eta(r_{\beta+1}) = \eta(v_{\beta+1}, r_{\beta+1}) = \beta$, for some $v_{\beta+1} \in V(S_{\beta+1})$ and for all limit ordinals $\beta < \alpha$ there is $\eta(S_\beta) = \eta(r_\beta) = \beta$.

Let $\alpha = \beta + 1$ be a successor ordinal. By construction, we see that $(S_{\alpha+1}, r_{\alpha+1})$ consists of the two copies of $(S_{\beta+1}, r_{\beta+1})$ on disjoint vertex sets, say (S, r) and (S', r') joined by the edge $\{r, r'\}$, with $r_{\alpha+1} = r$.

Consider the capture relation in $S_{\alpha+1}$. Note that every non-leaf vertex of a tree is a cut vertex; hence, the cop can forbid the robber to enter any component (that is, subtree) except the one occupied by the robber. Therefore, using the induction hypothesis, we have that $\eta(u, r) \le \beta$ for all $u \in V(S)$ and that there exists $v \in V(S')$ such that $\eta(v, r') = \beta$, while $\eta(u, r') \le \beta$ for all $u \in V(S')$. Since r' is a neighbour of r we have that $\eta(v, r) \le \beta + 1$ for any $v \in V(S_{\alpha+1})$, while $\eta(r', r) = \beta + 1$.

Observe that $\eta(r', v) \ge \eta(r', r)$ for any $v \in V(S)$ and, by the symmetry, for any $v \in V(S')$ we have that $\eta(r, v) \ge \eta(r, r') = \beta + 1$. Hence,

$$\eta(S_{\alpha+1}) = \eta(r) = \eta(r', r) = \beta + 1 = \alpha.$$

Now let α be a limit ordinal. Using the induction hypothesis it is straightforward to see that in $(S_\alpha, r_\alpha) = \bigoplus_{\beta<\alpha}(S_\beta, r_\beta)$, for any ordinal $\beta < \alpha$ there exists a neighbour of r_α, namely $r_{\beta+1}$, such that $\eta(r_{\beta+1}, r_\alpha) = \beta + 1$. On the other hand, for any $v \in V(S_\alpha) \setminus \{r_\alpha\}$ we have that $v \in V(S_\beta)$, for some $\beta < \alpha$; hence, there is $\eta(v, r) \le \beta + 1 < \alpha$. Therefore, we have that $\eta(r_\alpha) = \alpha$, while for any vertex $v \in V(S_\alpha)$ we have that $\eta(v, r_\alpha) < \alpha$. Moreover, we find that $\eta(r_\alpha, v) \ge \alpha$, for $v \in V(S_\alpha) \setminus \{r_\alpha\}$. This follows since a robber on r_α may choose to escape to any neighbour of r_α except at most the one leading towards vertex v, on which the cop resides. Hence, $\eta(S_\alpha) = \alpha = \eta(r_\alpha)$.

To finish this case consider tree $(S_{\alpha+1}, r_{\alpha+1})$ as two copies of (S_α, r_α) on disjoint vertex sets, say (S, r) and (S', r') joined by the edge $\{r, r'\}$, with $r_{\alpha+1} = r$. By considering the capture relation on $S_{\alpha+1}$, we see that $\eta(r', r) = \alpha$, while for any vertex $v \in V(S_{\alpha+1})$ there is $\eta(v, r) \le \alpha$. Of course there is $\eta(r', v) \ge \eta(r', r)$ for any $v \in V(S)$ and, by the symmetry, for any $v \in V(S')$ we have that $\eta(r, v) \ge \eta(r, r') = \alpha$.

Hence, we have that

$$\eta(S_{\alpha+1}) = \eta(r) = \eta(r', r) = \alpha,$$

and the proof follows. □

4. General cop-win graphs

Define

$$\Lambda = \{\rho(G) : G \text{ is a cop-win graph}\}.$$

Of course, $\Lambda_T \subseteq \Lambda$. However, the containment is strict as we describe in the following example.

Consider the following cop-win graph introduced in [16], which we refer to as the *Polat graph*. Let $X = \{x_n : n < \omega\}$, $Y = \{y_n : n < \omega\}$, and $Z = \{z\}$ be disjoint sets of vertices. Let G be the graph defined by $V(G) = X \cup Y \cup Z$ and

$$E(G) = \bigcup_{n<\omega}\Big(\{x_n, x_{n+1}\}, \{x_n, z\}, \{x_n, y_n\}, \{x_n, y_{n+1}\}, \{x_n, y_{n+2}\}, \{x_n, y_{n+3}\}\Big).$$

The Polat graph is suggested in Fig. 4. By direct checking, we have that $\rho(G) = \omega + 1$, $\eta(G) = \omega$ and $\theta(G) = X \cup Z$ (we omit the details here). Hence, the Polat graph witnesses the fact that $\omega + 1 \in \Lambda \setminus \Lambda_T$.

We may modify the construction of the Polat graph in several ways to obtain other CR-ordinals. For example, we may add a finite path to the vertex z obtaining graphs with CR-ordinal equalling $\omega + i$, for $0 < i < \omega$. By forming a sum (akin to the rooted tree sum described in the previous section) of these generalized Polat graphs, for $j < \omega, j > 1$, there exist cop-win graphs with CR-ordinal $\omega \cdot j + (i + j)$.

5. Problems

The main open problem we consider is to classify which ordinals belong to Λ. Define

$$\Upsilon = \left\{\omega \cdot i + (i + j) : i, j < \omega\right\} \cup \left\{\alpha + \omega : \alpha \text{ is a limit ordinal}\right\}.$$

Our family of cop-win graphs derived from the Polat graph supports the assertion that $\Lambda = \Upsilon$. We leave this as an open problem. Some ordinals do not seem possible to attain as a CR-ordinal. For example, is $\omega \in \Lambda$? Note that this question is answered negatively if $\Lambda = \Upsilon$. Observe that $\rho(G)$ is well-defined for any (not necessarily cop-win) graph. Another question, therefore, is to classify the ordinals $\rho(G)$.

We mention in closing that the paper [15] does not distinguish between cardinals and ordinals (though its results are correct when the distinction is made). Further, we do not know the best upper bound on the tower of relations $\{\leq_\alpha\}_{\alpha < \aleph_{\gamma+1}}$ for a graph G of cardinality $\aleph_\gamma$.

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