L(2,1)-labelling of Graphs*

Frédéric Havet† Bruce Reed†‡ Jean-Sébastien Sereni§

Abstract

An $L(2,1)$-labelling of a graph is a function f from the vertex set to the positive integers such that $|f(x) - f(y)| \geq 2$ if $\mathrm{dist}(x,y) = 1$ and $|f(x) - f(y)| \geq 1$ if $\mathrm{dist}(x,y) = 2$, where $\mathrm{dist}(x,y)$ is the distance between the two vertices x and y in the graph G. The *span* of an $L(2,1)$-labelling f is the difference between the largest and the smallest labels used by f plus 1. In 1992, Griggs and Yeh conjectured that every graph with maximum degree $\Delta \geq 2$ has an $L(2,1)$-labelling with span at most $\Delta^2 + 1$. By settling this conjecture for Δ sufficiently large, we prove the existence of a constant C such that the span of any graph of maximum degree Δ is at most $\Delta^2 + C$.

1 Introduction

In the channel assignment problem, transmitters at various nodes within a geographic territory must be assigned channels or frequencies in such a way as to avoid interferences. A model for the channel assignment problem developed wherein channels or frequencies are represented with integers, "close" transmitters must be assigned different integers and "very close" transmitters must be assigned integers that differ by at least 2. This quantification led to the definition of an $L(p,q)$-labelling of a graph $G = (V,E)$ as a function f from the vertex set to the positive integers such that $|f(x) - f(y)| \geq p$ if $\mathrm{dist}(x,y) = 1$ and $|f(x) - f(y)| \geq q$ if $\mathrm{dist}(x,y) = 2$, where $\mathrm{dist}(x,y)$ is the distance between the two vertices x and y in the graph G. The notion of $L(2,1)$-labelling first appeared in 1992 [6]. Since then, a large number of articles has been published devoted to the study of $L(p,q)$-labellings. We refer the interested reader to the surveys of Calamoneri [1] and Yeh [11].

Generalisations of $L(p,q)$-labellings in which for each $i \geq 1$, a minimum gap of p_i is required for channels assigned to vertices at distance i, have also been studied (see for example the recent survey of Griggs and Král' [5]).

In the context of the channel assignment problem, the main goal is to minimise the number of channels used. Hence, we are interested in the *span* of an $L(p,q)$-labelling f, which is the difference between the largest and the smallest labels of f plus 1. The $\lambda_{p,q}$*-number* of G is $\lambda_{p,q}(G)$, the minimum span over all $L(p,q)$-labellings of G. In general, determining the $\lambda_{p,q}$-number of a graph is NP-hard [3]. In their seminal paper, Griggs and Yeh [6] observed that a greedy algorithm yields $\lambda_{2,1}(G) \leq \Delta^2 + 2\Delta + 1$, where Δ is the maximum degree of the graph G. Moreover, they conjectured that this upper bound can be decreased to $\Delta^2 + 1$.

Conjecture 1.1. ([6]) *For every $\Delta \geq 2$ and every graph G of maximum degree Δ,*

$$\lambda_{2,1}(G) \leq \Delta^2 + 1.$$

This upper bound would be tight: there are graphs with degree Δ, diameter 2 and $\Delta^2 + 1$ vertices, namely the 5-cycle, the Petersen graph and the Hoffman-Singleton graph. Thus, their square is a clique of order $\Delta^2 + 1$, so the span of every $L(2,1)$-labelling is at least $\Delta^2 + 1$.

Jonas [7] improved slightly on Griggs and Yeh's upper bound by showing that every graph of maximum degree Δ admits a $(2,1)$-labelling with span at most $\Delta^2 + 2\Delta - 3$. Subsequently, Chang and Kuo [2] provided the upper bound $\Delta^2 + \Delta + 1$ which remained the best general upper bound for about a decade. Král' and Škrekovski [8] brought this upper bound down by 1 as the corollary of a more general result. And, using the algorithm of Chang and Kuo [2], Gonçalves [4] decreased this bound by 1 again, thereby obtaining the upper bound $\Delta^2 + \Delta - 1$. We prove the following approximate version of Conjecture 1.1.

Theorem 1.1. *There exists a constant C such that for every integer Δ and every graph of maximum degree Δ,*

$$\lambda_{2,1}(G) \leq \Delta^2 + C.$$

This result is obtained by combining any of the previously mentionned upper bounds with the next theorem, which settles Conjecture 1.1 for sufficiently large Δ.

*This work is partially supported by the European project IST FET AEOLUS and the ÉGIDE ECO-NET project 16305SB. The first and second author thank the Department of Applied Mathematics KAM-DIMATIA and the Institute for Theoretical Computer Science (ITI) of Charles University for their hospitality during their stay in Prague, when part of this research was conducted.

†Projet Mascotte, I3S (CNRS/UNSA)-INRIA, Sophia Antipolis, France.

‡School of Computer Science, McGill University, Montreal, Québec, Canada.

§Institute for Theoretical Computer Science ITI, and Department of Applied Mathematics KAM, Charles University, Prague, Czech Republic.

THEOREM 1.2. *There is a Δ_0 such that for every graph G of maximum degree $\Delta \geq \Delta_0$,*

$$\lambda_{2,1}(G) \leq \Delta^2 + 1.$$

Actually, we consider a more general setup. We are given a graph G_1 with vertex-set V, along with a spanning subgraph G_2. We want to assign integers from $\{1, 2, \ldots, k\}$ to the elements of V so that vertices adjacent in G_1 receive different colours and vertices adjacent in G_2 receive colours which differ by at least 2. Typically the maximum degree of G_1 is much larger than the maximum degree of G_2. In the case of $L(2,1)$-labelling, G_1 is the square of G_2. We impose the condition that for some integer Δ, G_1 has maximum degree at most Δ^2 and G_2 has maximum degree Δ. We show that under these conditions there exists a colouring for $k = \Delta^2 + 1$ provided that Δ is large enough. This is best possible since G_1 may be a clique of size $\Delta^2 + 1$.

THEOREM 1.3. *There is a Δ_0, such that for every $\Delta \geq \Delta_0$, and $G_2 \subseteq G_1$ with $\Delta(G_1) \leq \Delta^2$ and $\Delta(G_2) \leq \Delta$, there exists a $(\Delta^2 + 1)$-colouring c of $V(G_1)$ such that no edge of G_1 is monochromatic and for every edge $xy \in E(G_2)$, $|c(x) - c(y)| \geq 2$.*

In the next section we give an outline of the proof. We use G_1-*neighbour* to mean a neighbour in G_1 and G_2-*neighbour* to indicate a neighbour in G_2. For every vertex v and every subgraph H of G_1, we let $\deg^1_H(v)$ be the number of G_1-neighbours of v in H. We omit the subscript if $H = G_1$.

We do not explicit the value of Δ_0, and we assume that it is large enough so that the inequalities stated in the sequel hold.

2 A Sketch of the Proof

We consider a counter-example to Theorem 1.3 chosen so as to minimise V. Thus, for every proper subset X of the vertices of G_1, there is a $(\Delta^2 + 1)$-colouring c of X such that every edge of G_1 within X is non-monochromatic, and for every edge xy of G_2 contained within X, $|c(x) - c(y)| \geq 2$. Such a colouring of X is a *good colouring*. In particular, as $G_2 \subseteq G_1$, this implies that every vertex v has more than $\Delta^2 - 2\Delta$ G_1-neighbours as otherwise we could complete a good colouring of $V(G_1) - v$ greedily. Indeed for each vertex, a coloured G_2-neighbour forbids 3 colours, which is 2 more as being only a G_1-neighbour. The next lemma follows by setting $d = 1000\Delta$ and applying to G_1 a decomposition result due to Reed [9, Lemma 15.2].

LEMMA 2.1. *There is a partition of V into disjoint sets $D_1, \ldots, D_\ell, S$ such that*

- *(a) every D_i has between $\Delta^2 - 8000\Delta$ and $\Delta^2 + 4000\Delta$ vertices;*
- *(b) there are at most $8000\Delta^3$ edges of G_1 leaving any D_i;*
- *(c) a vertex has at least $\frac{3}{4}\Delta^2$ G_1-neighbours in D_i if and only if it is in D_i; and*
- *(d) for each vertex v of S, the neighbourhood of v in G_1 contains at most $\binom{\Delta^2}{2} - 1000\Delta^3$ edges.*

We let H_i be the subgraph of G_1 induced by D_i and $\overline{H_i}$ its complementary graph. An *internal neighbour* of a vertex of D_i is a neighbour in H_i. An *external neighbour* of a vertex of D_i is a neighbour that is not internal. The proof of the following lemma can be found in the full-length version of this article [10].

LEMMA 2.2. *For every i, $\overline{H_i}$ has no matching of size at least $10^3\Delta$.*

For each $i \in \{1, 2, \ldots, \ell\}$, we let M_i be a maximum matching of $\overline{H_i}$, and K_i be the clique $D_i - V(M_i)$. By Lemmas 2.1(a) and 2.2, $|K_i| \geq \Delta^2 - 10^4\Delta$. We let B_i be the set of vertices in K_i that have more than $\Delta^{5/4}$ external G_1-neighbours, and we set $A_i := K_i \setminus B_i$. Considering Lemma 2.1(b) we make the following observation.

FACT 2.1. *For every index $i \in \{1, 2, \ldots, \ell\}$, $|B_i| \leq 8000\Delta^{7/4}$ and so $|A_i| \geq \Delta^2 - 9000\Delta^{7/4}$.*

We are going to colour the vertices in three steps. We first colour $V_1 := V \setminus \cup_{i=1}^{\ell} A_i$ except some vertices of S. Then we colour the vertices of $V_2 := \cup_{i=1}^{\ell} A_i$. We finish by colouring the uncoloured vertices of S greedily.

In order to extend the (partial) colouring of V_1 to V_2 and to finish the colouring of the vertices of S, we need some properties. We will prove the following.

LEMMA 2.3. *There is a good colouring c of a subset Y of V_1 such that*

- *(i) every uncoloured vertex of V_1 is in S;*
- *(ii) for each edge xy of every M_i, $c(x) = c(y)$;*
- *(iii) for every uncoloured vertex v of V_1 there are at least 2Δ colours that appear on two G_1-neighbours of v; and*
- *(iv) for every colour j and clique A_i there are at most $\frac{4}{5}\Delta^2$ vertices of A_i that have either a G_1-neighbour outside D_i coloured j or a G_2-neighbour outside D_i coloured using $j - 1, j$ or $j + 1$.*

We then show that a colouring that verifies the conditions of Lemma 2.3 can be extended to $Y \cup V_2$.

Lemma 2.4. *Every good colouring of a subset Y of V_1 satisfying conditions (i)–(iv) of Lemma 2.3 can be completed to a good colouring of $Y \cup V_2$.*

By Lemma 2.3(*iii*), we can then complete the colouring by colouring the vertices of $V_1 - Y$ greedily. Thus to prove our theorem, we need only prove Lemmas 2.3 and 2.4, which we do in the next two sections. We use several probabilistic tools, namely the Lovász Local Lemma, the Chernoff Bound, Talagrand's and McDiarmid's Inequalities. Each of these tools is presented in the book of Molloy and Reed [9], and most are presented in many other places. Each omitted proof can be found in the full-length version of this article [10].

3 The Proof of Lemma 2.3

In this section, we want to find a good colouring for an appropriate subset Y of $G[V_1]$, which satisfies conditions (*i*)–(*iv*) of Lemma 2.3. We actually construct new graphs G_1^* and G_2^* and consider good colourings of these graphs. This will help us to ensure that the conditions of Lemma 2.3 hold.

3.1 Forming G_1^* and G_2^* For $j \in \{1, 2\}$, we obtain G'_j from G_j by contracting each edge of each M_i into a vertex (that is, we consider these vertex pairs one by one, replacing the pair xy with a vertex adjacent to all of the neighbours of both x and y in the graph). We let C_i be the set of vertices obtained by contracting the pairs in M_i. We set $V^* := V_1 - \cup_{i=1}^{\ell} V(M_i) + \cup_{i=1}^{\ell} C_i$. For each $i \in \{1, 2, \dots, \ell\}$, let Big_i be the set of vertices of V^* not in $B_i \cup C_i$ that have more than $\Delta^{9/5}$ neighbours in A_i. We construct G_1^* from G'_1 by removing the vertices of $\cup_{i=1}^{\ell} A_i$ and adding for each i an edge between every pair of vertices in Big_i. And G_2^* is obtained from G'_2 by removing the vertices of $\cup_{i=1}^{\ell} A_i$.

Note that $G_2^* \subseteq G_1^*$. Our aim is to colour the vertices of V^* except some of S such that vertices adjacent in G_1^* are assigned different colours, and vertices adjacent in G_2^* are assigned colours at distance at least 2. Such a colouring is said to be *nice*. To every partial nice colouring of V^* is associated the good colouring of V_1 obtained as follows: each coloured vertex of $V \cap V^*$ keeps its colour, and for each index i, every pair of matched vertices of M_i is assigned the colour of the corresponding vertex of C_i. So this partial good colouring satisfies condition (*ii*) of Lemma 2.3.

Definition 3.1. For every vertex u and every subset F of V^*,

- the number of G_1^*-neighbours of u in F is $\delta_F^1(u)$;
- the number of G_2^*-neighbours of u in F is $\delta_F^2(u)$; and
- $\delta_F^*(u) := \delta_F^1(u) + 2\delta_F^2(u)$.

For all these notations, we omit the subscript if $F = V^*$.

The next lemma bounds these parameters.

Lemma 3.1. *Let v be a vertex of V^*. The following hold.*

(*i*) $\delta^2(v) \le 2\Delta$, *and if* $v \notin \cup_{i=1}^{\ell} C_i$ *then* $\delta^2(v) \le \Delta$;

(*ii*) *if* $v \in S \cap \mathrm{Big}_i$ *for some* i, *then* $\delta^1(v) \le \Delta^2 - 8\Delta$;

(*iii*) $\delta^1(v) \le \Delta^2$, *and if* $v \notin S$ *then* $\delta^1(v) \le \frac{3}{4}\Delta^2$.

Proof. (*i*) To obtain G_2^*, we only removed some vertices and contracted some pairwise disjoint pairs of non-adjacent vertices. Consequently, the degree of each new vertex is at most twice the maximum degree of G_2, i.e. 2Δ, and the degree of the other vertices is at most their degree is G_2, hence at most Δ.

(*ii*) By Lemma 2.1(*b*), we have $|\mathrm{Big}_i| \le 8000\Delta^{6/5}$ for each index i. Moreover, a vertex v can be in Big_i for at most $\Delta^{1/5}$ values of i. Recall that for each index i such that $v \in S \cap \mathrm{Big}_i$, the vertex v has at least $\Delta^{9/5}$ G_1-neighbours in A_i. So, in the process of constructing G_1^*, it looses at least $\Delta^{9/5}$ edges and gain at most $8000\Delta^{7/5}$ edges. Consequently, the assertion follows because $\Delta^{9/5} \ge 8000\Delta^{7/5} + 8\Delta$.

(*iii*) By (*ii*), if $v \in S$ then $\delta^1(v) \le \deg^1(v) \le \Delta^2$. Assume now that $v \notin S$, hence $v \in B_i \cup C_i$ for some index i. By Lemma 2.2, each set C_i has at most 1000Δ vertices and by Fact 2.1, each set B_i has at most $8000\Delta^{7/4}$ vertices. Moreover, by Lemma 2.1(*c*), each vertex of D_i has at most $\frac{1}{4}\Delta^2$ G_1-neighbours outside of D_i. It follows that each vertex of $B_i \cup C_i$ has at most $\frac{1}{2}\Delta^2 + 1000\Delta + 8000\Delta^{7/4} + 8000\Delta^{7/5} \le \frac{3}{4}\Delta^2$ G_1^*-neighbours.

Our construction of G'_1 and G'_2 is designed to deal with condition (*ii*) of Lemma 2.3. The edges we add between vertices of Big_i are designed to help with condition (*iv*). The bound of $\frac{3}{4}\Delta^2$ on the degree of the vertices of $V^* \setminus S$ in the last lemma, helps us to ensure that condition (*i*) holds.

To ensure that condition (*iii*) holds, we would like to use condition (*i*) and the fact that sparse vertices have many non-adjacent pairs of G_1-neighbours. However, in constructing G_1^*, we contracted some pairs of non-adjacent vertices and added edges between some other pairs of non-adjacent vertices. As a result, possibly some vertices in S are no longer sparse. We have to treat such vertices carefully.

We define $\hat{S}$ to be those vertices in S that have at least 90Δ neighbours outside S. Then $\hat{S}$ contains all the vertices which may no longer be sufficiently sparse, as we note next.

LEMMA 3.2. *Each vertex of $S \setminus \hat{S}$ has at least $450\Delta^3$ pairs of G_1-neighbours in S that are not adjacent in G_1^*.*

Proof. Let $s \in S \setminus \hat{S}$. We know that s has at least $\Delta^2 - 2\Delta$ G_1-neighbours. Hence it has at least $\binom{\Delta^2}{2} - 4\Delta^3$ pairs of G_1-neighbours. Thus, by Lemma 2.1(d), s has at least $996\Delta^3$ pairs of G_1-neighbours that are not adjacent in G_1. Since $s \notin \hat{S}$, all but at most $90\Delta^3$ such pairs lie in $N(s) \cap S$. Let Ω be the collection of pairs of G_1-neighbours of s in S that are not adjacent in G_1. Then $|\Omega| \geq 906\Delta^3$. For convenience, we say that a pair of Ω is *suitable* if its vertices are not adjacent in G_1^*.

Let s_1 be a member of a pair of Ω. If s_1 does not belong to $\cup_{i=1}^{\ell} \mathrm{Big}_i$, then every vertex of S that is not adjacent to s_1 in G_1 is also not adjacent to s_1 in G_1^*. Thus every pair of Ω containing s_1 is suitable.

If $s_1 \in \cup_{i=1}^{\ell} \mathrm{Big}_i$, then for each index i such that $s_1 \in \mathrm{Big}_i$, the vertex s_1 has at least $\Delta^{9/5}$ G_1-neighbours in A_i. Hence, there are at least $\Delta^2 - 92\Delta - (\Delta^2 - \Delta^{9/5}) = \Delta^{9/5} - 92\Delta$ pairs of Ω containing s_1. Recall from the proof of Lemma 3.1 that the number of edges added to s_1 by the construction of G_1^* is at most $8000\Delta^{7/5} < \frac{1}{2}\Delta^{9/5} - 46\Delta$. Consequently, the number of suitable pairs of Ω containing the vertex s_1 is at least half the number of pairs of Ω containing s_1.

Therefore, we conclude that at least $\frac{1}{2}|\Omega| > 450\Delta^3$ pairs of Ω are suitable.

It turns out that we will colour all of $\hat{S}$, which makes it easier to ensure that condition (*iii*) holds.

3.2 High Level Overview Our first step is to colour some of S, including all of $\hat{S}$. We do this in two phases. In the first one, we consider assigning each vertex of S a colour at random. We show by analysing this random procedure that there is a partial nice colouring of S such that every vertex of $S \setminus \hat{S}$ satisfies condition (*iii*) of Lemma 2.3. In the second phase, we finish colouring the vertices of $\hat{S}$. We use an iterative quasi-random procedure. In each iteration but the last, each vertex chooses a colour, from those which do not yield a conflict with any already coloured neighbour, uniformily at random. The last iteration has a similar flavour.

We then turn to colouring the vertices in the sets B_i and C_i. Our degree bounds imply that we could do this greedily. However, we will mimic the iterative approach just discussed. We use this complicated colouring process because it allows us to ensure that condition (*iv*) of Lemma 2.3 holds for the colouring we obtain. At any point during the colouring process, $\mathrm{Notbig}_{i,j}$ is the set of vertices $v \in A_i$ such that v has either a G_1'-neighbour $u \notin \mathrm{Big}_i \cup D_i$ that has colour j or a G_2'-neighbour $u \notin \mathrm{Big}_i \cup D_i$ that has colour $j-1, j$ or $j+1$. The challenge is to construct a colouring such that $\mathrm{Notbig}_{i,j}$ remains small for every index i and every colour j.

3.3 Colouring Sparse Vertices As mentioned earlier, we colour sparse vertices in two phases. The first one provides a partial nice colouring of S satisfying condition (*iii*) of Lemma 2.3. The second one extends this nice colouring to all the vertices of $\hat{S}$, using an iterative quasi-random procedure.

We will need a lemma to bound the size of $\mathrm{Notbig}_{i,j}$. We consider the following setting. We have a collection of at most Δ^2 subsets of vertices. Each set contains at most Q vertices, and no vertex lies in more than $\Delta^{9/5}$ sets. A random experiment is conducted, where each vertex is marked with probability at most $\frac{1}{Q\cdot\Delta^{2/5}}$. We moreover assume that, for any set of $s \geq 1$ vertices, the probability that all are marked is at most $\left(\frac{1}{Q\cdot\Delta^{2/5}}\right)^s$. Note that this is in particular the case if the vertices are marked independently.

LEMMA 3.3. *Under the preceding hypothesis, the probability that at least $\Delta^{37/20}$ sets contain a marked vertex is at most* $\exp\left(-\Delta^{1/20}\right)$.

Proof. For every $i \in \{1, 2, \ldots, 9\}$, let E_i be the event that at least $\frac{1}{9}\Delta^{37/20}$ sets contain a marked member of T_i, where T_i is the set of vertices lying in between $\Delta^{(i-1)/5}$ and $\Delta^{i/5}$ sets. Note that if at least $\Delta^{37/20}$ sets contain at least one marked vertex, then at least one the events E_i must hold.

The total number of vertices in the sets being at most $\Delta^2 Q$, we deduce that $|T_i| \leq \frac{\Delta^2 Q}{\Delta^{(i-1)/5}}$. Furthermore, if E_i holds then at least $\frac{1}{9}\Delta^{37/20-i/5}$ vertices of T_i must be marked. Therefore,

$$\begin{aligned}
\mathbf{Pr}(E_i) &\leq \binom{\Delta^2 Q/\Delta^{(i-1)/5}}{\frac{1}{9}\Delta^{37/20-i/5}} \cdot \left(\frac{1}{Q\Delta^{2/5}}\right)^{\frac{1}{9}\Delta^{37/20-i/5}} \\
&\leq \left(\frac{e\Delta^2 Q/\Delta^{(i-1)/5}}{\frac{1}{9}\Delta^{37/20-i/5} \times Q\Delta^{2/5}}\right)^{\frac{1}{9}\Delta^{37/20-i/5}} \\
&\leq \left(\frac{9e}{\Delta^{1/20}}\right)^{\frac{1}{9}\Delta^{37/20-i/5}}.
\end{aligned}$$

Since $\frac{1}{9}\Delta^{37/20-i/5} \geq \frac{1}{9}\Delta^{1/20}$, the probability that E_i holds is at most $\frac{1}{9}\exp\left(-\Delta^{1/20}\right)$, and therefore the sought result follows.

3.3.1 First Step

Lemma 3.4. *There exists a nice colouring of a subset H of S with colours in $\{1, 2, \ldots, \Delta^2 + 1\}$ such that*

(i) every uncoloured vertex v of $S \setminus \hat{S}$ has at least 2Δ colours appearing at least twice in $N_S(v) := N_{G_1}(v) \cap S$;

(ii) every vertex of S has at most $\frac{19}{20}\Delta^2$ coloured G_1^*-neighbours;

(iii) for every index i and every colour j, the size of $\mathrm{Notbig}_{i,j}$ is at most $\Delta^{19/10}$.

Proof. For convenience, let us set $C := \Delta^2 + 1$. We use the following colouring procedure.

1. Each vertex of S is activated with probability $\frac{9}{10}$.
2. Each activated vertex is assigned a colour of $\{1, 2, \ldots, C\}$, independently and uniformly at random.
3. A vertex which gets a colour creating a conflict — i.e. assigned to one of its G_1^*-neighbours, or at distance less than 2 of a colour assigned to one of its G_2^*-neighbours — is uncoloured.

We aim at applying the Lovász Local Lemma to prove that, with positive probability, the resulting colouring fulfils the three conditions of the lemma. Let v be a vertex of G. We let $E_1(v)$ be the event that v does not fulfil condition (i), and $E_2(v)$ be the event that v does not fulfil condition (ii). For each i, j, let $E_3(i, j)$ be the event that the size of $\mathrm{Notbig}_{i,j}$ exceeds $\Delta^{19/10}$. It suffices to prove that each of those events occurs with probability less than Δ^{-17}. Indeed, each event is mutually independent of all events involving vertices or dense sets at distance more than 4 in G_1^* or G_1'. Moreover, each vertex of any set A_i has at most $\Delta^{5/4}$ external neighbours in G, and $|A_i| \le \Delta^2 + 1$. Thus, each event is mutually independent of all but at most Δ^{16} other events. Consequently, the Lovász Local Lemma applies since $\Delta^{-17} \times \Delta^{16} < \frac{1}{4}$, and yields the sought result.

Hence, it only remains to prove that the probability of each event is at most Δ^{-17}. Let us start with $E_2(v)$. We define W to be the number of activated neighbours of v. Thus, $\mathbf{Pr}(E_2(v)) \le \mathbf{Pr}\left(W > \frac{19}{20}\Delta^2\right)$. We set $m := |N(v) \cap S|$, and we may assume that $m > \frac{19}{20}\Delta^2$. The random variable W is a binomial on m variables with probability $\frac{9}{10}$. In particular, its expected value $\mathbf{E}(W)$ is $\frac{9m}{10}$. Applying the Chernoff Bound to W with $t = \frac{m}{20}$, we obtain that

$$\begin{aligned}\mathbf{Pr}\left(W > \tfrac{19}{20}\Delta^2\right) &\le \mathbf{Pr}\left(|W - \mathbf{E}(W)| > \tfrac{m}{20}\right)\\ &\le 2\exp\left(-\tfrac{m^2 \cdot 10}{400 \cdot 27m}\right) \le \Delta^{-17},\end{aligned}$$

since $\frac{19}{20}\Delta^2 < m \le \Delta^2$.

Let $v \in S \setminus \hat{S}$. We now bound $\mathbf{Pr}(E_1(v))$. By Lemma 3.2, let Ω be a collection of $450\Delta^3$ pairs of G_1-neighbours of v in S that are not adjacent in G_1^*. We consider the random variable X defined as the number of pairs of Ω whose members (i) are both assigned the same colour j, (ii) both retain that colour, and (iii) are the only two vertices in $N(v)$ that are assigned j. Thus, X is at most the number of colours appearing at least twice in $N_S(v)$. The probability that some non-adjacent pair of vertices u, w in $N(v)$ satisfies (i) is $\frac{9}{10} \cdot \frac{9}{10} \cdot \frac{1}{C}$. In total, the number of G_1^*-neighbours of v, u, w in H is at most $3\Delta^2$, and the number of G_2^*-neighbours of u and w is at most 4Δ. Therefore, given that they satisfy (i), the vertices u and w also satisfy (ii) and (iii) with probability at least $\left(1 - \frac{1}{C}\right)^{3\Delta^2} \cdot \left(1 - \frac{2}{C}\right)^{4\Delta}$. Consequently,

$$\mathbf{E}(X) \ge 450\Delta^3 \cdot \frac{81}{100C} \exp\left(-\frac{3\Delta^2}{C}\right) \exp\left(-\frac{8\Delta}{C}\right) > 3\Delta.$$

Hence, if $E_1(v)$ holds then X must be smaller than its expected value by at least Δ. But we assert that

$$\mathbf{Pr}\left(\mathbf{E}(X) - X > \Delta\right) \le \Delta^{-17}, \tag{3.1}$$

which will yield the desired result.

To establish Equation (3.1), we apply Talagrand's Inequality. We set X_1 to be the number of colours assigned to at least two vertices in $N(v)$, including both members of at least one pair in Ω, and X_2 is the number of colours that (i) are assigned to both members of at least one pair in Ω, and (ii) create a conflict with one of their neighbours, or are also assigned to at least one other vertex in $N(v)$. Note that $X = X_1 - X_2$. Therefore, by what precedes, if $E_1(v)$ holds then either X_1 or X_2 must differ from its expected value by at least $\frac{1}{2}\Delta$. Notice that

$$\mathbf{E}(X_2) \le \mathbf{E}(X_1) \le C \cdot 450\Delta^3 \cdot \frac{1}{C^2} \le 450\Delta.$$

If $X_1 \ge t$, then there is a set of at most $4t$ trials whose outcomes certify this, namely the activation and colour assignment for t pairs of variables. Moreover, changing the outcome of any random trial can only affect X_1 by at most 2, since it can only affect whether the old colour and the new colour are counted or not. Thus Talangrand's Inequality applies and, since $\mathbf{E}(X_1) \ge \mathbf{E}(X) > 3\Delta$, we obtain that

$$\begin{aligned}\mathbf{Pr}\left(|X_1 - \mathbf{E}(X_1)| > \frac{1}{2}\Delta\right) &\le 4\exp\left(-\frac{\Delta^2}{32 \cdot 64 \cdot 450\Delta}\right)\\ &\le \frac{1}{2}\Delta^{-17}.\end{aligned}$$

Similarly, if $X_2 \geq t$ then there is a set of at most $6t$ trials whose outcomes certify this fact, namely the activation and colour assignment of t pairs of vertices and, for each of these pairs, the activation and colour assignment of a colour creating a conflict to a neighbour of a vertex of the pair. As previously, changing the outcome of any random trial can only affect X_2 by at most 2. Therefore by Talagrand's Inequality, if $\mathbf{E}(X_2) \geq \frac{1}{2}\Delta$ then

$$\mathbf{Pr}\left(|X_2 - \mathbf{E}(X_2)| > \frac{1}{2}\Delta\right) \leq 4\exp\left(-\frac{\Delta^2}{32 \cdot 96 \cdot 450\Delta}\right) \leq \frac{1}{2}\Delta^{-17}.$$

If $\mathbf{E}(X_2) < \frac{1}{2}\Delta$, then we consider a binomial random variable that counts each vertex of $N_S(v)$ independently with probability $\frac{1}{4|N_S(v)|}\Delta$. We let X'_2 be the sum of this random variable and X_2. Note that $\frac{1}{4}\Delta \leq \mathbf{E}(X'_2) \leq \frac{3}{4}\Delta$ by Linearity of Expectation. Moreover, observe that if $|X_2 - \mathbf{E}(X_2)| > \frac{1}{2}\Delta$ then $|X'_2 - \mathbf{E}(X'_2)| > \frac{1}{4}\Delta$. Therefore, by applying Talagrand's Inequality to X'_2 with $c = 2, r = 6$ and $t = \frac{1}{4}\Delta \in [60c\sqrt{r\,\mathbf{E}(X'_2)}, \mathbf{E}(X'_2)]$, we also obtain in this case that

$$\begin{aligned}\mathbf{Pr}\left(|X_2 - \mathbf{E}(X_2)| > \frac{1}{2}\Delta\right) &\leq \mathbf{Pr}\left(|X'_2 - \mathbf{E}(X'_2)| > \frac{1}{4}\Delta\right)\\ &\leq 4\exp\left(-\frac{\Delta^2}{16 \cdot 192 \cdot 3 \cdot \Delta}\right)\\ &\leq \frac{1}{2}\Delta^{-17}.\end{aligned}$$

Consequently, we infer that $\mathbf{Pr}(\mathbf{E}(X) - X > \Delta) \leq \Delta^{-17}$, as desired.

It only remains now to deal with $E_3(i,j)$. We use Lemma 3.3. For each i, every vertex of A_i has at most $\Delta^{5/4}$ external neighbours. Moreover, for each colour j, each such neighbour is activated and assigned a colour in $\{j-1, j, j+1\}$ with probability at most $\frac{9}{10} \cdot \frac{3}{C} < \frac{1}{\Delta^{5/4} \cdot \Delta^{2/5}}$. As these assignments are made independently, the conditions of Lemma 3.3 are fulfilled, so we deduce that the probability that $E_3(i,j)$ holds is at most $\exp\left(-\Delta^{1/20}\right) \leq \Delta^{-17}$. Thus, we obtained the desired upper bound on $\mathbf{Pr}(E_3(i,j))$, which concludes the proof.

3.3.2 Second Step In the second step, we extend the partial colouring of S to all the vertices of $\hat{S}$. To do so, we need the following general lemma, that will also be used in the next subsection to colour the vertices of the sets $B_i \cup C_i$. Its proof is long and technical, and we omit it (the reader may consult the associated research report [10] for details).

LEMMA 3.5. *Let F be a subset of V^* with a partial nice colouring, and H be a set of uncoloured vertices of F. For each vertex u of H, let $L(u)$ be the colours available to colour u, that is that create no conflict with the already coloured vertices of $F \cup H$. We assume that for every vertex u, $|L(u)| \geq 16\Delta^{33/20}$ and $|L(u)| \geq \delta^1_H(u) + 6\Delta$.*
Then, the partial nice colouring of F can be extended to a nice colouring of H such that for every index $i \in \{1, 2, \ldots, \ell\}$ and every colour j, the size of $\mathrm{Notbig}_{i,j}$ *increases by at most $\Delta^{19/10}$.*

Consider a partial nice colouring of S obtained in the first step. In particular, $|\mathrm{Notbig}_{i,j}| \leq \Delta^{19/10}$. We wish to ensure that every vertex of $\hat{S}$ is coloured. This can be done greedily, but to be able to continue the proof we need to have more control on the colouring. We shall apply Lemma 3.5 to the set H of uncoloured vertices in $\hat{S}$. For each vertex $u \in H$, the list $L(u)$ is initialised as the list of colours that can be assigned to u without creating any conflict. By Lemmas 3.1 and 3.4(ii), $|L(u)| \geq \frac{1}{20}\Delta^2 - 4\Delta \geq 16\Delta^{33/20}$.

Suppose that u is in no set Big_i. Then $\delta^1_S(u) \leq \deg^1_S(u) \leq \Delta^2 - 90\Delta$, and u has at most Δ G^*_2-neighbours. Hence, we infer that $|L(u)| \geq \delta^1_H(u) + 88\Delta$. Assume now that u belongs to some set Big_i. By Lemma 3.1(i) and (ii), we have $\delta^1(u) \leq \Delta^2 - 8\Delta$ and $\delta^2(u) \leq \Delta$. So, $|L(u)| \geq \delta^1_H(u) + 8\Delta - 2\Delta = \delta^1_H(u) + 6\Delta$.

Therefore, by Lemma 3.5 we can extend the partial nice colouring of S to $\hat{S}$ such that $|\mathrm{Notbig}_{i,j}| \leq 2\Delta^{19/10}$ for every index i and every colour j.

3.4 Colouring the Sets B_i and C_i Let $H := \bigcup_{i=1}^{\ell}(B_i \cup C_i)$. At this stage, the vertices of H are uncoloured. We first apply Lemma 3.5 to extend the partial nice colouring of S to the vertices of H in such a way that $\mathrm{Notbig}_{i,j}$ does not grow too much, for every index i and colour j. Next, we will show that the good colouring derived from this nice colouring satisfies the conditions of Lemma 2.3.

For each vertex u of H, let $L(u)$ be the lists of colours that would not create any conflict with the already coloured vertices. By Lemma 3.1(iii), $\delta^1(u) \leq \frac{3}{4}\Delta^2$. Hence, $|L(u)| \geq \frac{1}{4}\Delta^2 + \delta^1_H(u) - 4\Delta \geq \max\left(16\Delta^{33/20}, \delta^1_H(u) + 6\Delta\right)$.

Therefore, by Lemma 3.5, we extend the partial nice colouring of the vertices of S to the vertices of $\bigcup_{i=1}^{\ell}(B_i \cup C_i)$. Moreover, for each index i and each colour j, the size of each $\mathrm{Notbig}_{i,j}$ is at most $3\Delta^{19/10}$.

Consider now the partial good colouring of V_1 associated to this nice colouring. Let us show that it satisfies the conditions of Lemma 2.3. By the definition, it satisfies conditions (i) and (ii). Condition (iii) follows from Lemma 3.4. Hence, it only remains to show that condition (iv) holds.

Fix an index i and a colour j. Recall that Big_i is a clique, so there is at most one vertex of Big_i of each colour. Consequently, the number of vertices of A_i with a G_1-neighbour in Big_i coloured j is at most $\max\left(2 \cdot \frac{1}{4}\Delta^2, \frac{3}{4}\Delta^2\right) = \frac{3}{4}\Delta^2$, by Lemma 2.1$(c)$. Besides, the number of vertices of A_i with a G_2-neighbour in Big_i coloured $j-1$ or $j+1$ is at most 4Δ. Finally, the number of vertices of A_i with either a G_1-neighbour not in $\mathrm{Big}_i \cup D_i$ coloured j, or a G_2-neighbour not in $\mathrm{Big}_i \cup D_i$ coloured $j-1, j$ or $j+1$ is at most $|\,\mathrm{Notbig}_{i,j}\,| \leq 3\Delta^{19/10}$. Thus, all together, the number of vertices of A_i with a G_1-neighbour not in $B_i \cup C_i$ coloured j, or a G_2-neighbour not in $B_i \cup C_i$ coloured $j-1$ or $j+1$ is at most

$$\frac{3}{4}\Delta^2 + 3\Delta^{19/10} + 4\Delta \leq \frac{4}{5}\Delta^2.$$

This concludes the proof of Lemma 2.3.

4 The Proof of Lemma 2.4

We consider a good colouring of V satisfying the conditions of Lemma 2.3. The procedure we apply is comprised of two phases. In the first phase, a random permutation of colours is assigned to the vertices of A_i. In doing so, we might create two kinds of conflicts: a vertex of A_i coloured j might have an external G_1-neighbour coloured j, or a G_2-neighbour coloured $j-1$ or $j+1$. We shall deal with these conflicts in a second phase. To be able to do so, we first ensure that the colouring obtained in the first phase fulfils some properties.

Proposition 4.1.

$$|A_i| + |B_i| + \frac{1}{2}|V(M_i)| \leq \Delta^2 + 1.$$

Proof. By the maximality of M_i, for every edge $e = xy$ of M_i there is at most one vertex v_e of K_i that is adjacent to both x and y in $\overline{H_i}$. Hence, every edge e of M_i has an endvertex $n(e)$ that is adjacent in H_i to every vertex of K_i except possibly one, called $x(e)$. By Lemmas 2.1 and 2.2,

$$|A_i| + |B_i| \geq \Delta^2 - 8000\Delta - 2.10^3\Delta \geq 10^3\Delta > |M_i|.$$

So there exists a vertex $v \in A_i \cup B_i \setminus \cup_{e \in M_i} x(e)$. The vertex v is adjacent in G_1 to all the vertices of K_i (except itself) and all the vertices $n(e)$ for $e \in M_i$. So

$$|A_i| + |B_i| - 1 + \frac{1}{2}|V(M_i)| \leq \deg^1(v) \leq \Delta^2.$$

4.1 Phase 1 For each set A_i, we choose a subset of $a_i := |A_i|$ colours as follows. First, we exclude all the colours that appear on the vertices of $B_i \cup C_i$. Moreover, if a colour j is assigned to at least three pairs of vertices matched by M_i, not only do we exclude the colour j but also the colours $j-1$ and $j+1$. By Proposition 4.1 and because every edge of M_i is monochromatic by Lemma 2.3(ii), we infer that at least a_i colours have not been excluded. Then we assign a random permutation of those colours to the vertices of A_i. We let Temp_i be the subset of vertices of A_i with an external G_1-neighbour of the same colour, or a G_2-neighbour with a colour at distance less than 2.

Lemma 4.1. *With positive probability, the following hold.*

- *(i) For each i, $|\,\mathrm{Temp}_i\,| \leq 3\Delta^{5/4}$;*
- *(ii) for each index i and each colour j, at most $\Delta^{19/10}$ vertices of A_i have a G_1-neighbour in $\cup_{k \neq i} A_k$ coloured j or a G_2-neighbour in $\cup_k A_k$ coloured $j-1$ or $j+1$.*

Proof. We use the Lovász Local Lemma. For every index i, we let $E_1(i)$ be the event that $|\,\mathrm{Temp}_i\,|$ is greater than $3\Delta^{5/4}$. For each index i and each colour j, we define $E_2(i,j)$ to be the event that condition (ii) is not fulfilled. Each event is mutually independent of all events involving dense sets at distance greater than 2, so each event is mutually independent of all but at most Δ^9 other events. According to the Lovász Local Lemma, it is enough to show that each event has probability at most Δ^{-10}, since $\Delta^9 \times \Delta^{-10} < \frac{1}{4}$.

Our first goal is to upper bound $\mathbf{Pr}(E_1(i))$. We may assume that both the colour assigments for all cliques other than A_i, and the choice of the a_i colours to be used on A_i have already been made. Thus it only remains to choose a random permutation of those a_i colours onto the vertices of A_i. Since every vertex $v \in A_i$ has at most $\Delta^{5/4}$ external neighbours and Δ G_2-neighbours, the probability that $v \in \mathrm{Temp}_i$ is at most $(\Delta^{5/4} + 4\Delta)/a_i$. So we deduce that $\mathbf{E}(|\,\mathrm{Temp}_i\,|) \leq \Delta^{5/4} + 4\Delta$. We define a binomial random variable B that counts each vertex of A_i independently with probability $\Delta^{5/4}/(2a_i)$. We set $X := |\,\mathrm{Temp}_i\,| + B$. By Linearity of Expectation,

$$\frac{1}{2}\Delta^{5/4} \leq \mathbf{E}(X) = \mathbf{E}(|\,\mathrm{Temp}_i\,|) + \frac{1}{2}\Delta^{5/4} \leq 2\Delta^{5/4}.$$

Moreover, if $|\,\mathrm{Temp}_i\,| > 3\Delta^{5/4}$ then $|\,\mathrm{Temp}_i\,| - \mathbf{E}(|\,\mathrm{Temp}_i\,|) > \Delta^{5/4}$, and hence $X - \mathbf{E}(X) > \frac{1}{2}\Delta^{5/4}$. We now apply McDiarmid's Inequality to show that X is concentrated. Note that if $|\,\mathrm{Temp}_i\,| \geq s$, then the colours to $2s$ vertices (that is, s members of Temp_i

and one neighbour for each) certify that fact. Moreover, switching the colours of two vertices in A_i may only affect whether those two vertices are in Temp_i, and whether at most four vertices with a colour at distance less than 2 are in Temp_i. So we may apply McDiarmid's Inequality to X with $c = 6, r = 2$ and $t = \frac{1}{2}\Delta^{5/4} \in \left[60c\sqrt{r\,\mathbf{E}(X)}, \mathbf{E}(X)\right]$. We deduce that the probability that the event $E_1(i)$ holds is at most

$$\begin{aligned}&\mathbf{Pr}\left(|X - \mathbf{E}(X)| > \frac{1}{2}\Delta^{5/4}\right)\\ <&4\exp\left(-\frac{\Delta^{5/2}}{4\times 32\times 36\times 2\Delta^{5/4}}\right)\\ <&\Delta^{-10}.\end{aligned}$$

We now upper bound $\mathbf{Pr}(E_2(i,j))$. To this end, we use Lemma 3.3. Recall that the vertices of A_i get different colours. Every vertex $v \in A_i$ has at most $\Delta^{5/4}$ external neighbours, and Δ G_2-neighbours. We set $Q := \Delta^{5/4} + \Delta$. We let $S(v)$ be the set of all vertices that are either external G_1-neighbours of v, or G_2-neighbours of v. Hence, $|S(v)| \le Q$. Note that each vertex is in at most $\Delta^{5/4}$ sets $S(v)$ for $v \in A_i$. Each vertex of a set $S(v)$ is assigned a colour in $\{j-1, j, j+1\}$ with probability at most

$$\max_k \frac{3}{a_k} < \frac{1}{3Q\times\Delta^{2/5}},$$

because $\min a_k \ge \Delta^2 - 9000\Delta^{7/4}$ by Fact 2.1. Moreover, at most three vertices in each set A_k are assigned a colour in $\{j-1, j, j+1\}$. As the random permutations for different cliques are independent, Lemma 3.3 implies that the probability that more than $\Delta^{37/20}$ vertices of A_i have an external G_1-neighbour in some A_k coloured j, or a G_2-neighbour in some A_k coloured $j-1, j$ or $j+1$ is at most $\exp\left(-\Delta^{1/20}\right) < \Delta^{-10}$. This concludes the proof.

4.2 Phase 2 We consider a colouring γ satisfying the conditions of Lemma 4.1. For each set A_i and each vertex $v \in \mathrm{Temp}_i$ we let $\mathrm{Swappable}_v$ be the set of vertices u such that

(a) $u \in A_i \setminus \mathrm{Temp}_i$;

(b) $\gamma(u)$ does not appear on an external G_1-neighbour of v;

(c) $\gamma(v)$ does not appear on an external G_1-neighbour of u;

(d) $\gamma(u)-1$ and $\gamma(u)+1$ do not appear on a G_2-neighbour of v;

(e) $\gamma(v)-1$ and $\gamma(v)+1$ do not appear on a G_2-neighbour of u.

LEMMA 4.2. *For every $v \in \mathrm{Temp}_i$, the set* $\mathrm{Swappable}_v$ *contains at least $\frac{1}{10}\Delta^2$ vertices.*

Proof. Let us upper bound the number of vertices that are not in $\mathrm{Swappable}_v$. By Lemma 4.1(i), at most $3\Delta^{5/4}$ vertices of A_i violate condition (a) and at most $\Delta^{5/4}$ vertices violate condition (b) by the definition of A_i. As v has at most Δ G_2-neighbours, the number of vertices violating condition (d) is at most 2Δ. According to Lemma 2.3(iv), the number of vertices of A_i violating conditions (c) or (e) because of a neighbour not in $\left(\cup_{k=1}^{\ell} A_k\right) \cup (B_i \cup C_i)$ is at most $\frac{4}{5}\Delta^2$. Moreover, by the way we chose the a_i colours for A_i, the number of vertices violating condition (e) because of a neighbour in $B_i \cup C_i$ is at most 10Δ. Finally, the number of vertices violating conditions (c) or (e) because of a colour assigned during Phase 1 is at most $\Delta^{19/10}$ thanks to Lemma 4.1(ii). Therefore, we deduce that the size of $\mathrm{Swappable}_v$ is at least

$$|A_i| - \frac{4}{5}\Delta^2 - \Delta^{19/10} - 4\Delta^{5/4} - 12\Delta - 1 \ge \frac{1}{10}\Delta^2,$$

as $|A_i| \ge \Delta^2 - 9000\Delta^{\frac{7}{4}}$ by Fact 2.1.

For each index i and each vertex $v \in \mathrm{Temp}_i$, we choose 100 uniformly random members of $\mathrm{Swappable}_v$. These vertices are called *candidates* of v.

DEFINITION 4.1. A candidate u of v is *unkind* if either

(a) u is a candidate for some other vertex;

(b) v has an external neighbour w that has a candidate w' with the same colour as u;

(c) v has a G_2-neighbour w that has a candidate w' coloured $\gamma(u)-1, \gamma(u)$ or $\gamma(u)+1$;

(d) v has an external neighbour w that is a candidate for exactly one vertex w', with $\gamma(w') = \gamma(u)$;

(e) v has a G_2-neighbour w that is a candidate for exactly one vertex w', that is coloured $\gamma(u)-1, \gamma(u)$ or $\gamma(u)+1$;

(f) u has an external neighbour w that has a candidate w' with the same colour as v;

(g) u has a G_2-neighbour w that has a candidate w' coloured $\gamma(v)-1, \gamma(v)$ or $\gamma(v)+1$;

(h) u has an external neighbour w that is a candidate for a vertex w' with the same colour as v; or

(i) u has a G_2-neighbour w that is a candidate for a vertex w' coloured $\gamma(v)-1, \gamma(v)$ or $\gamma(v)+1$.

A candidate of v is *kind* if it is not unkind.

LEMMA 4.3. *With positive probability, for each index i, every vertex of* Temp_i *has a kind candidate.*

We choose candidates satisfying the preceding lemma. For each vertex $v \in \mathrm{Temp}_i$ we swap the colour of v and one of its kind candidates. The obtained colouring is the desired one. So to conclude the proof of Lemma 2.4, it only remains to prove Lemma 4.3.

Proof of Lemma 4.3. For every vertex v in some Temp_i, let $E_1(v)$ be the event that v does not have a kind candidate. Each event is mutually independent of all events involving dense sets at distance greater than 2. So each event is mutually independent of all but at most Δ^9 other events. Hence, we shall prove that the probability of each event is at most Δ^{-10}, and so the conclusion will follow from the Lovász Local Lemma since $\Delta^{-10} \cdot \Delta^9 < \frac{1}{4}$.

Observe that the probability that a particular vertex of $\mathrm{Swappable}_v$ is chosen is $100/|\mathrm{Swappable}_v|$, which is at most $1000\Delta^{-2}$.

We wish to upper bound $\mathbf{Pr}(E_1(v))$ for an arbitrary vertex $v \in \mathrm{Temp}_i$, so we can assume that all vertices but v have already chosen candidates. By Lemma 4.1(i), the number of vertices that satisfy condition (a) of Definition 4.1 is at most $300\Delta^{5/4}$. Note that the vertex v has at most $\Delta^{5/4}$ external neighbours, each having at most 100 candidates. Since each colour appears on at most one member of $\mathrm{Swappable}_v$, we deduce that the number of vertices satisfying one of the conditions (b) and (d) is at most $101\Delta^{5/4}$. Similarly, the number of vertices satisfying one of the conditions (c) and (e) is at most 303Δ.

We deal now with the remaining four conditions, starting with condition (f). The number of vertices of A_i that satisfy condition (f) is at most the number of edges with an endvertex in A_i and an endvertex in A_k with $k \neq i$, and such that the external endvertex has chosen a candidate with the colour of v. For each vertex $w \in \cup_{k\neq i} A_k$, we let N_w be the number of G_1-neighbours of w in A_i. So, $N_w \leq \Delta^{5/4}$. Note that $\sum N_w \leq 8000\Delta^3$ by Lemma 2.1(b). We define the random variable F_w to be N_w if w has a candidate with the colour of v, and 0 otherwise. Thus, the number of vertices of A_i that satisfy condition (f) is at most the sum σ of the variables F_w for $w \in \cup_{k\neq i} A_k$. We aim at showing that

$$\mathbf{Pr}\left(\sigma > 2\Delta^{3/2}\right) < \frac{1}{8}\Delta^{-10}. \tag{4.2}$$

Since each vertex in some set Temp_k chooses its candidates independently, the variables F_w are independent. For each $r \in \{0, 1, \dots, \lceil \log_2 (\Delta^{5/4}) \rceil\}$, let S_r be the set of vertices w of $\cup_{k\neq i} A_k$ such that $2^{r-1} < N_w \leq 2^r$. So,

$$\sigma \leq \sum_{r=0}^{\lceil \log_2(\Delta^{5/4})\rceil} \sum_{w\in S_r} F_w \leq \sum_{r=0}^{\lceil \log_2(\Delta^{5/4})\rceil} 2^r \sigma_r$$

where $\sigma_r := |\{w \in S_r : F_w \neq 0\}|$. Consequently, to prove (4.2) it suffices to show that for every index r,

$$\mathbf{Pr}(\sigma_r > t) < \frac{\Delta^{-10}}{8\left(\lceil \log_2(\Delta^{5/4})\rceil + 1\right)}$$

where $t := \frac{2\Delta^{3/2}}{2^r\left(\lceil \log_2(\Delta^{5/4})\rceil+1\right)}$.

Fix an index r. As the variables F_w are independent, the probability that σ_r is more than t is no more than the probability that the binomial random variable $\mathrm{BIN}(n,p)$ with $n := \frac{8000}{2^{r-1}}\Delta^3$ and $p := 1000\Delta^{-2}$ is more than t. Therefore, we deduce from Chernoff's Bound that

$$\begin{aligned}\mathbf{Pr}(\sigma_r > t) &\leq \mathbf{Pr}\left(\mathrm{BIN}(n,p) - np > \frac{t}{2}\right)\\ &< 2\exp\left(\frac{t}{2} - \left(np + \frac{t}{2}\right)\ln\left(1 + \frac{t}{2np}\right)\right)\\ &< \frac{\Delta^{-10}}{8\left(\lceil \log_2(\Delta^{5/4})\rceil + 1\right)},\end{aligned}$$

as wanted.

A similar argument shows that, with probability at least $1 - \frac{1}{8}\Delta^{-10}$, at most $2\Delta^{3/2}$ vertices of A_i satisfy condition (g).

We consider now condition (h). A vertex u of A_i satisfies condition (h) if it has an external G_1-neighbour that was chosen as a canditate for a vertex with the same colour as v. We actually consider the number of edges with an endvertex in A_i and the other in some A_k with $k \neq i$, and such that the endvertex not in A_i is a candidate for a vertex with the same colour as v. We express this as the sum of several random variables.

Recall that N_w is the number of G_1-neighbours of w in A_i, for every $w \in \cup_{k\neq i} A_k$. So, $N_w \leq \Delta^{5/4}$. We define X_w to be N_w if w is a candidate for a vertex with the colour of v, and 0 otherwise. Thus, the probability that $X_w = N_w$ is at most $1000\Delta^{-2}$. The number of vertices of A_i satisfying condition (h) is at most the sum τ of the variables X_w for $w \in \cup_{k\neq i} A_k$. Our aim is to show that

$$\mathbf{Pr}\left(\tau > 2\Delta^{3/2}\right) < \frac{1}{8}\Delta^{-10}. \tag{4.3}$$

Recall that

$$S_r = \{w \in \cup_{k\neq i} A_k : 2^{r-1} < N_w \leq 2^r\}$$

for every $r \in \{0, 1, \ldots, \lceil \log_2 (\Delta^{5/4}) \rceil\}$. Hence,

$$\tau \leq \sum_{r=0}^{\lceil \log_2 (\Delta^{5/4}) \rceil} \sum_{w \in S_r} X_w \leq \sum_{r=0}^{\lceil \log_2 (\Delta^{5/4}) \rceil} 2^r \tau_r$$

where $\tau_r := |\{w \in S_r : X_w \neq 0\}|$. Consequently, to prove (4.3) it suffices to show that for every index r,

$$\mathbf{Pr}(\tau_r > t) < \frac{\Delta^{-10}}{8\left(\lceil \log_2 (\Delta^{5/4}) \rceil + 1\right)} \tag{4.4}$$

where $t := \frac{2\Delta^{3/2}}{2^r\left(\lceil \log_2 (\Delta^{5/4}) \rceil + 1\right)}$.

Let us fix an index r. Observe that τ_r is at most $100 \sum_{k \neq i} Z_r^k$ where each Z_r^k is a zero-one random variable, which is 1 if there is a vertex of $S_r \cap A_k$ that is a candidate for a vertex with the same colour as v, and 0 otherwise. In particular, $Z_r^k = 1$ with probability at most $1000|S_r \cap A_k|\Delta^{-2}$. Moreover, if $\tau_r > t$ then $\sum_{k \neq i} Z_r^k > \frac{t}{100}$. Let $R_r := 2^{1-r} \cdot 8000\Delta^3$. By Lemma 2.1($b$), for every $k \neq i$ the size of $S_r \cap A_k$ is at most $M_r := \min\left(\Delta^2, R_r\right)$. We set

$$T_m := \{k \neq i : 2^{m-1} \leq |S_r \cap A_k| \leq 2^m\}$$

for every integer $m \in \{0, 1, \ldots, \lceil \log_2(M_r) \rceil\}$. Hence, $|T_m| \leq 2^{2-m-r} \cdot 8000\Delta^3$, and

$$\tau_r \leq 100 \sum_{m=0}^{\lceil \log_2(M_r) \rceil} \sum_{k \in T_m} Z_r^k.$$

Let us fix an index m. The variables Z_r^k for $k \in T_m$ are independent zero-one random variables, each being 1 with probability at most $2^m \cdot 1000\Delta^{-2}$. Observe that if $2^m \geq \Delta^2/1000$, then $|T_m| \leq 32 \cdot 10^6 \cdot 2^{-r}\Delta$ and hence $\tau_r \leq t$ so that (4.4) holds. Thus we assume in the sequel that $2^m \leq \Delta^2/1000$. We define Y_m to be the sum of $2^{2-m-r} \cdot 8000\Delta^3$ independent zero-one random variables, each being 1 with probability $2^m \cdot 1000\Delta^{-2}$. Thus, $\sum_{k \in T_m} Z_r^k \leq Y_m$. The expected value of Y_m is

$$\mathbf{E}(Y_m) = 32 \cdot 10^6 \cdot 2^{-r}\Delta < \Delta^{3/2}.$$

Setting $t' := \frac{t}{100 \cdot (\lceil \log_2(M_r) \rceil + 1)}$, we deduce from Chernoff's Bound that

$$\begin{aligned}
&\mathbf{Pr}\left(Y_m - \mathbf{E}(Y_m) > \frac{t'}{2}\right) \\
<&2\exp\left(\frac{t'}{2} - \ln\left(1 + \frac{t'}{2\,\mathbf{E}(Y_m)}\right)(\mathbf{E}(Y_m) + \frac{t'}{2})\right) \\
<&\frac{\Delta^{-10}}{8\left(\lceil \log_2 (\Delta^{5/4}) \rceil + 1\right)\left(\lceil \log_2 (M_r) \rceil + 1\right)}.
\end{aligned}$$

This implies (4.4), which in turn implies (4.3), as desired.

A similar argument shows that the probability that more than $\Delta^{3/2} - 200\Delta$ vertices of A_i satisfy condition (i) because of an external G_2-neighbour is at most $\frac{1}{8}\Delta^{-10}$. Moreover, at most 200Δ vertices satisfy condition (i) because of an internal G_2-neighbour.

Therefore, with probability at least $1 - \frac{1}{2}\Delta^{-10}$ the number of unkind members of $\mathrm{Swappable}_v$ is at most

$$8\Delta^{3/2} + 300\Delta^{5/4} + 101\Delta^{5/4} + 303\Delta < \Delta^{7/4}.$$

In this case, the probability that no candidate is kind is at most

$$\left(\frac{\Delta^{7/4}}{\Delta^2/10}\right)^{100} < \frac{1}{2}\Delta^{-10}.$$

Consequently, the probability that $E_1(v)$ holds is at most $\frac{1}{2}\Delta^{-10} + \frac{1}{2}\Delta^{-10} = \Delta^{-10}$, as desired. This concludes the proof.

References

[1] T. Calamoneri. The $L(h,k)$-Labeling Problem: A Survey, (2004). Available from: `http://www.dsi.uniroma1.it/~calamo/survey.html`

[2] G. J. Chang and D. Kuo, The L(2, 1)-labeling problem on graphs, *SIAM J. Discr. Math.*, 9 (1996), pp. 309–316.

[3] J. P. Georges, D. W. Mauro and M. A. Whittlesey, Relating path coverings to vertex labelings with a condition at distance two, *Discrete Math.*, 135 (1994), pp. 103–111.

[4] D. Gonçalves, On the L(p,1)-labelling of graphs, *Discrete Math.*, to appear.

[5] J. R. Griggs and D. Král', Graph labellings with variable weights, a survey, *Submitted.*

[6] J. R. Griggs and R. K. Yeh, Labeling graphs with a condition at distance two, *SIAM J. Discr. Math.*, 11 (1992), pp. 585–595.

[7] T. K. Jonas, Graph coloring analogues with a condition at distance two: $L(2,1)$-labelings and list λ-labelings, *Ph.D. Thesis*, University of South Carolina, 1993.

[8] D. Král' and R. Škrekovski, A theorem about the channel assignment, *SIAM J. Discr. Math.*, 16 (2003), pp. 426–437.

[9] M. Molloy and B. Reed, *Graph colouring and the probabilistic method.* Algorithms and Combinatorics, 23. Springer-Verlag, Berlin, 2002.

[10] F. Havet, B. Reed and J.-S. Sereni, $L(2,1)$-labelling of graphs, *ITI-Series*. Available from: `http://kam.mff.cuni.cz/~sereni/`

[11] R. K. Yeh, A survey on labeling graphs with a condition at distance two, *Discrete Math.*, 306 (2006), no. 12, 1217–1231.