Exercise session ~~8~~ 7 – Prob. & Stat. 2 — Nov 14, 2023

Conditional expectation & Joint PDF

If $\mathbb{E}(X \mid Y = y) = f(y)$, then we define $\mathbb{E}(X \mid Y)$ as $f(Y)$. We have

$$\mathbb{E}(\mathbb{E}(X \mid Y)) = \mathbb{E}(X) \qquad \text{(law of iterated expectation)}.$$

Recall that if X and Y are continuous, then

$$\mathbb{E}(X \mid Y = y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx = \int_{-\infty}^{\infty} x \frac{f_{X,Y}(x,y)}{f_Y(y)} dx.$$

We also define $var(X|Y)$ as $g(Y)$, if $var(X|Y = y) = g(y)$. We have

$$var(X) = \mathbb{E}(var(X \mid Y)) + var(\mathbb{E}(X \mid Y)) \qquad \text{(law of total variance)}.$$

1. Warm-up: (a) what is $\mathbb{E}(X \mid X)$?
(b) what is $\mathbb{E}(X \mid Y)$, when X and Y are independent?

2. A gambler repeatedly takes part in a game, where with probability $p > 1/2$ he wins the amount he betted and with probability $1 - p$ he loses it. A popular strategy, known as Kelly strategy, is to always bet $2p - 1$ multiple of your current total. What is the expected fortune the gambler obtains after n rounds? (And why is it better then betting everything?)

3. Pat and Nat are dating, and all of their dates are scheduled to start at 9 p.m. Nat always arrives promptly at 9 p.m. Pat is highly disorganized and arrives at a time that is uniformly distributed between 8 p.m. and 10 p.m. Let X be the time in hours between 8 p.m. and the time when Pat arrives. If Pat arrives before 9 p.m., their date will last exactly 3 hours. If Pat arrives after 9 p.m., their date starts when Pat arrives and lasts for a time uniformly distributed between 0 and $3 - X$. Nat gets irritated when Pat is late and will end relationship after the second date where Pat is late by more than 45 minutes. All random variables mentioned are independent.
(a) What is the expected number of hours Nat waits for Pat to arrive?
(b) What is the expected duration of any particular date?
(c) What is the expected number of dates they will have before breaking up?

4. Show that for a discrete or continuous random variable X and any function $g(Y)$ of another (discrete) random variable Y, we have

$$\mathbb{E}(Xg(Y)|Y) = g(Y)\mathbb{E}(X \mid Y).$$

5. * Let X and Y be independent random variables. Use the law of total variance to show that

$$var(XY) = \mathbb{E}(X)^2 var(Y) + \mathbb{E}(Y)^2\, var(X) + var(X)\, var(Y).$$

6. * We toss n times a biased coin whose probability of heads, denoted by q, is the value of a random variable Q with given mean μ and positive variance σ^2. Let $X_1, \ldots, X_n$ be a Bernoulli random variable that models the outcome of the ith toss (i.e., $X_i = 1$ if the ith toss is a head). We assume that $X_1, \ldots, X_n$ are conditionally independent, given $Q = q$. Let X be the number of heads obtained in the n tosses.
(a) Use the law of iterated expectations to find $\mathbb{E}(X_i)$ and $\mathbb{E}(X)$.
(b) Find $cov(X_i, X_j)$. Are $X_1, \ldots, X_n$ independent?
(c) Use the law of total variance to find $var(X)$. Verify your answer using the covariance result of part (b).

7. Let (X, Y) have a joint pdf that is uniform on a triangle with vertices $(0,0)$, $(0,1)$, $(1,0)$.

(a) Find the joint pdf.
(b) Find marginal PDF of Y.
(c) Find $f_{X|Y}$.
(d) Find $\mathbb{E}(X \mid Y = y)$. Use total expectation law to find $\mathbb{E}(X)$ in terms of $\mathbb{E}(Y)$.
(e) Find $\mathbb{E}(X)$, $\mathbb{E}(Y)$.
(f) Find $\mathbb{E}(X \mid Y)$.

8. Choosing a point uniformly on the sphere is equivalent to choosing the longitude λ uniformly from $[-\pi, \pi]$ and choosing the latitude φ from $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with density $\frac{1}{2}\cos\varphi$.

(a) Think about why this is so.
(b) What is $f(\lambda|\varphi = 0)$?
(c) What is $f(\varphi|\lambda = 0)$?
(d) Is this strange or not? (Possibly yes, it is called Borel-Kolmogorov paradox.)