Exercise session 1 – Prob. & Stat. 2 — Oct 3, 2023

Markov chains basics

Recall that a sequence $(X_t)_{t=0}^{\infty}$ of random variables with range S is a (discrete time, time-homogeneous) *Markov chain* if for every $t \geq 0$ and every $a_0, \dots, a_t = i, a_{t+1} = j \in S$ we have

$$P(X_{t+1} = j \mid X_t = i \,\&\, X_{t-1} = a_{t-1} \,\&\, \dots \,\&\, X_0 = a_0) = P(X_{t+1} = j \mid X_t = i) = p_{i,j},$$

for some collection of transition probabilities $p_{i,j}$. The condition is only required when the conditional probabilities are defined, that is when $P(X_t = a_t \,\&\, \dots \,\&\, X_0 = a_0) > 0$.

1. A Markov chain with states $\{1, 2, 3\}$ has transition matrix

$$\begin{pmatrix} .4 & .4 & .2 \\ 0 & .8 & .2 \\ .9 & 0 & .1 \end{pmatrix}$$

Draw its transition graph.

2. Consider the Markov chain from the previous problem.

(a) Find $P(X_4 = 3 \mid X_3 = 2)$.
(b) Find $P(X_3 = 1 \mid X_2 = 3)$.
(c) Suppose $P(X_0 = 1) = 0.2$. Find $P(X_0 = 1 \,\&\, X_1 = 2)$.
(d) Suppose $P(X_0 = 1) = 0.2$. Find $P(X_0 = 1 \,\&\, X_1 = 2 \,\&\, X_2 = 3)$.
(e) Suppose that $X_0 = 1$. What is $P(X_3 = 1)$?

3. Show that any sequence of independent identically distributed random variables taking values in a countable set S is a Markov chain. What if the variables are independent, but each may have a different distribution?

4. Let us modify the example with broken/working machine from the class: If the machine is broken, the probability that it will be repaired in another day is still 0.9. However, if it is broken for the second day in a row, the probability that it will be repaired is only 0.5. If the machine is broken for three days in a row, it is broken forever.

(a) Can you represent this with a Markov chain?
(b) Suppose the probability that a working machine breaks increases to 0.1 after a year (starting from the last successful repair). Is this a Markov chain?

5. A mouse moves along a tiled corridor with $2m$ tiles, where $m > 1$. From each tile $i \neq 1, 2m$ it moves to either tile $i-1$ or $i+1$ with equal probability. From tile 1 or tile $2m$, it moves to tile 2 or $2m-1$, respectively, with probability 1. Each time the mouse moves to a tile $i \leq m$ or $i > m$, an electronic device outputs a signal L or R, respectively.

(a) Is the position of the mouse a Markov chain?
(b) Can the generated sequence of signals L and R be described as a Markov chain with states L and R?

6. Consider the Markov chain that describes fly's movement (in the example from the lecture). Assume that the process starts at any of the four states, with equal probability. Let $Y_n = 1$ whenever the Markov chain is at state 0 or 1, and $Y_n = 2$ whenever the Markov chain is at state 2 or 3. Is the process Y_n a Markov chain?

7. A die is rolled repeatedly. Which of the following are Markov chains? For those that are, supply the transition graph.

(a) The largest number M_n shown up to the n-th roll.
(b) The number N_n of sixes in n rolls.
(c) At time r, the time A_r after the most recent six.
(d) At time r, the time B_r before the next six.

8. Suppose (X_t) is a Markov chain. Show that $(X_{2t})_{t=0}^{\infty}$ is also a Markov chain. What is its transition matrix?

A bit of theory

9. Below is a formal proof of the transition probabilities for several steps. In the computation below, where exactly are we using the fact we are dealing with a Markov chain? Can you explain each of the equations?

$$P(X_{k+1} = j \mid X_0 = i) = \sum_{\ell=1}^{s} P(X_{k+1} = j \,\&\, X_k = \ell \mid X_0 = i) \tag{1}$$

$$= \sum_{\ell=1}^{s} P(X_{k+1} = j \mid X_k = \ell \,\&\, X_0 = i) P(X_k = \ell \mid X_0 = i) \tag{2}$$

$$= \sum_{\ell=1}^{s} P(X_{k+1} = j \mid X_k = \ell) P(X_k = \ell \mid X_0 = i) \tag{3}$$

10. (Unsuprising, but used all the time) Given a time-homogeneous Markov chain (X_t) and $i, j \in S$, we have

$$P(X_t = j \mid X_0 = i) = P(X_{k+t} = j \mid X_k = i).$$

Why is it true?

11. Are the following statements equivalent?

1. $P(\exists t > 0 : X_t = j \mid X_0 = i) > 0$
2. $\exists t > 0 : P(X_t = j \mid X_0 = i) > 0$