Combinatorics and graph theory 3 – 2020/21

Series 5

1. Show that every graph G has a tree decomposition of width $tw(G)$ where no two parts are in inclusion.

   Tree decomposition $(T, (V_t)_{t \in T})$ is *smooth*), if for some k

   - $|V_t| = k + 1$ for all $t \in V(T)$ and
   - $|V_s \cap V_t| = k$ for all edges $st \in E(T)$

2. Show that every graph G has a smooth tree decomposition of width $tw(G)$.

3. If $(T, (V_t)_{t \in T})$ is a smooth tree decomposition then $|V(T)| = |V(G)| - k$. In particular $|V(T)| \leq |V(G)|$.

For a graph G with a given weight $w : V(G) \to \mathbb{R}+$, an (k, α)-*separator* is a set $X \subseteq V(G)$ such that $|X| \leq k$ and for each component K of $G - X$ we have $w(K) \leq \alpha w(G)$.

Here $w(K) = \sum_{v \in V(K)} w(v)$. For unweighted graphs consider $w \equiv 1$.

4. Show that every weighted tree has a $(1, \frac{1}{2})$-separator. Show how to find it.

5. Show that every graph of tree width k has $(k + 1, \frac{1}{2})$-separator, and we can find it (given a tree decomposition) find it in linear time.

---

6. Given a graph G and its tree decomposition of width k. Decide if G is 3-colorable in time $O^*(3^k)$ – a shortcut for $O(3^k |V(G)|^l)$ some constant l.

7. Given a graph G and its tree decomposition of width k. Find its minimal vertex cover in time $O^*(2^k)$.