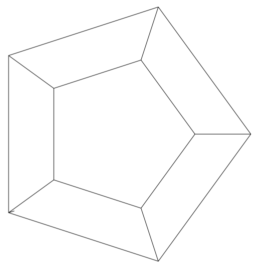
Combinatorics and graph theory 3 – 2020/21
Series 4

1. Find the tree width of C_n and $K_{n,n}$.

2. Find the tree width of graph of regular tetrahedron, cube, and octahedron.

3. Find the tree width of Wagner graph W_8 and the fivesided prizm (on the figure).



4. Show that $n \times n$ grid has tree-width at most n.

5. Show that $n \times n$ grid has tree-width exactly n.

6. Show that every planar graph is a minor of some grid.

7. Observe, that a graph can be obtained by a clique-sum of its torsos.

8. Can a tree-width go down by subdividing an edge? Can it go up?

9. Suppose G has a simplicial decomposition into k-colorable parts. Then G itself is k-colorable.

10. ($\star$) Prove that if G has tree-width at most k, then G has a tree decomposition $(T, \mathcal{V})$ of width at most k such that $|V(T)| \leq n$.

11. ($\star$) A graph G is *outerplanar* if it can be drawn in plane so that every vertex of G is incident with the outer face. Prove that every outerplanar graph has tree-width at most 2.

12. Let G be a graph T a set and $(V_t)_{t \in T}$ a collection of sets satisfying T1 and T2 from the definition of tree decomposition. Show, that there is a tree on T for which T3 is also true IF AND ONLY IF we can write $T = \{t_1, t_2, \ldots, t_n\}$ so that for every $2 \leq k \leq n$ there is $j < k$ satisfying

$$V_{t_k} \cap \bigcup_{i<k} V_{t_i} \subseteq V_{t_j}.$$

(The new condition is frequently easier to verify.)

13. *A separation* of a graph G is a pair (U_1, U_2) of graphs such that $G = U_1 \cup U_2$. Separations (U_1, U_2) and (W_1, W_2) are *compatible* if there are $i, j \in \{1, 2\}$ such that $U_i \subseteq W_j$ and $U_{3-i} \supseteq W_{3-j}$. Show that separations $S_e = (U_1, U_2)$ from the class are (for different choices of tree edges) compatible.

    (Harder bonus:) On the other hand, for every system $\mathcal{S}$ of compatible graph separations there is a tree decomposition $(T, \mathcal{V})$ for which $\mathcal{S} = \{S_e : e \in T\}$.