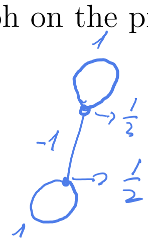
Combinatorics and Graph Theory III - 2020/21

Series 12

1. Let $\vec{C}_n$ denote the directed n-cycle, $\vec{P}_n$ the directed path on n vertices, and $\vec{K}_n$ the transitive tournament on n vertices. Define a *gain* of a cycle/path in a digraph as the difference between forward and backward edges (it is only defined upto changing a sign). Show that
   (a) $G \hom \vec{C}_n$ iff the gain of every cycle in G is divisible by n
   (b) $G \hom \vec{P}_n$ iff the gain of every cycle in G is 0 and the gain of every path is at most $n-1$
   (c) $G \hom \vec{K}_n$ iff not $\vec{P}_{n+1} \to G$
2. Show that we can decide in polynomial time whether there is a homomorphism $G \to \vec{C}_n$, $G \to \vec{P}_n$, and $G \to \vec{K}_n$.
3. Let G be the graph on the picture. Show that $hom(F, G)$ is the number of such sets of edges that cover every note.

4. Let G be the graph on the picture. Show that $hom(F, G)$ is 1 if F is Eulerian and 0 otherwise.



5. Let K be the graph with a single note of weight 1 and a loop of weight 1/2. For a random graph $G = G(n, 1/2)$ we have $\delta_\square(G, K) = o(1)$ with high probability.
6. Let G_1, G_2 be two simple graphs with $\delta_\square(G_1, G_2) = 0$. Show that there is a simple graph G and integers $n_1, n_2 \geq 1$ such that $G_i \cong G(n_i)$.
7. Let H be the graph with two nonadjacent vertices with a loop at each of them. Show that $\hat{\delta}_\square(H, K_2) = 1/4$, but $\delta_\square(H, K_2) = 1/8$.
8. Show that if n is odd then $\hat{\delta}_\square(K_{n,n}, \bar{K}_{n,n}) > \delta_\square(K_{n,n}, \bar{K}_{n,n})$.