Combinatorics and Graph Theory III - 2020/21

Series 10

1. Let $A = (a_{i,j})$ be an $n \times n$ matrix over a field F and suppose its permanent nonzero (over F). Then for any vector $b = (b_1, \dots, b_n) \in F^n$ and for any family of sets S_1, $\dots$, S_n of F, each of cardinality 2, there is a vector $x \in S_1 \times \dots \times S_n$, such that Ax differs from b in every coordinate.

2. Find a connected graph G with an orientation with all in-degrees 2, coefficient of $x_1^2 x_2^2 \dots x_n^2$ in its polynomial is 0, but still G is 3-choosable.

3. Let v_1, $\dots$, v_n be an ordering of vertices of G such that $v_1 v_2 \in E(G)$ and for $i \geq 3$ vertex v_i has exactly two neighbors among $\{v_1, \dots, v_{i-1}\}$ Let

$$p_G = \prod_{v_i v_j \in E(G), i<j} (x_j - x_i)$$

be the graph polynomial of G in variables x_1, $\dots$, x_n. For any function $f : [n] \to N$ let $c(f, g)$ be the coefficient of $x_1^{2-f(1)} x_2^{2-f(2)} \dots x_n^{2-f(n)}$ in p_G. (If $f(x) > 2$ for some $x \in [n]$ then $c(f, G) = 0$.) Let $e_i : [n] \to N$ be a function such that $e_i(i) = 1$ and $e_i(x) = 0$ for $x \in [n] \setminus \{i\}$. Write $\widetilde{c}(f, G) = c(f + e_2, G) - c(f + e_1, G)$.

   Show by induction over $n \geq 2$, that if f satisfies $\sum_{x \in [n]} f(x) = 2$, then $\widetilde{c}(f, G) \equiv 1 \pmod 3$.

4. Let G be 2-degenerated, $v_0 \in V(G)$, L a list assignments such that $|L(v)| \geq 3$ for $v \in V(G) \setminus \{v_0\}$ and $|L(v_0)| = 1$. Show that G is L-colorable.

5. Let G be a graph with a Hamilton cycle K such that $G - E(K)$ is a collection of vertex-disjoint triangles. Show that G is 3-choosable.